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Optimizing Manufacturing Technology: Unraveling Symmetry in Cubic Equation Roots

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In the realm of engineering the quest for optimization is ceaseless. This article explores the intricate relationship between cubic equations and the practical world of production technologies, unearthing the profound connections that underpin mathematical symmetry and its role in engineering. Cubic equations, often arising in the analysis of mechanical systems, electric circuits, and robotics, serve as indispensable tools for understanding and enhancing real-world applications. This study delves into the methods for finding the roots of cubic equations, shedding light on the vital role of mathematics in engineering and manufacturing technology.

Keywords: Symmetry, Cubic equations, Symmetric polynomials, Manufacturing Technology, Optimization, Wolfram Mathematics

1 Introduction

Symmetry, a concept both profound and ubiquitous, reverberates throughout the natural world and resonates in the tapestry of human culture. Its applications span the spectrum from practical, as observed in both manufacturing technology and mechanical engineering, to the purely theoretical realms of mathematics, where symmetry serves as the foundation for an array of theoretical fields. Notably, symmetry plays a pivotal role in the study of algebraic equations, particularly cubic equations, which offer a compelling bridge between the theoretical and practical worlds.

Cubic equations, along with their algebraic counterparts of higher degrees, serve as powerful tools in exploring linear systems' behaviors, revealing natural patterns that emerge in diverse contexts. These patterns, often manifesting as vibration frequencies or complex dynamic systems, are unveiled as the roots of algebraic equations corresponding to the ordinary differential equations governing the systems' dynamics. The degree of these algebraic equations depends on the number of linear components within the system, leading to intriguing scenarios such as cubic equations arising in the analysis of systems with three capacitors or in trajectory generation profiles for lower limb exoskeleton robots. [1, 2]

Within the field of complex numbers, an algebraically closed algebraic structure, every algebraic equation is guaranteed to possess at least one complex root. This intrinsic feature is intricately connected to the fundamental principles of symmetric polynomials, as evidenced by Vieta's formulas, an innovation from the 16th century. In contrast, the 19th century bore witness to the groundbreaking work of Henrik Abel and Évariste Galois, who established that for general polynomials of degree greater than four, no formulas can express their roots using basic arithmetic operations and nth roots (radicals). This revelation laid the cornerstone for Galois theory and group theory, major branches of abstract algebra. [3, 4]

But what is the relevance of these intricate mathematical explorations to the fields of engineering and manufacturing technology? The significance becomes evident when we consider that cubic equations and their ilk frequently play an indispensable role in solving practical problems in these domains. In engineering, for instance, the behavior of mechanical systems, electric circuits, or even the motion of robotics often leads to cubic equations, as the dynamics of these systems can be elegantly described and analyzed using mathematical models that rely on cubic polynomials. The insights gained from understanding the roots of these equations enable engineers to optimize designs, improve performance, and make informed decisions that impact the real-world applications of these technologies. [5-8]

This article embarks on a journey into the world of cubic equations, with a focus on discerning the methods for finding their roots in radicals. It explores the intricate relationship between the quest for roots and the realm of basic symmetric polynomials. To simplify these intricate calculations, we employ the power of Wolfram Mathematica, ensuring that the focus remains on the essence of the connection between root formulas and symmetric polynomials. [9]

Our primary objective in this article is to assert the existence of a formula for cubic equation roots and, in doing so, underscore the critical role of mathematics in engineering and manufacturing technology. While the well-known quadratic formula effectively indicates the rationality of its solutions, the cubic formula, in contrast, does not offer this clarity. In many engineering applications, especially cubic equations of state, rational roots are paramount. Therefore, the general cubic formula may not be ideally suited for practical purposes, leading to the use of numerical methods for approximating rational roots.

The true accomplishment of this paper, however, is not the discovery of a formula for cubic equation roots. Instead, we aim to illuminate the key to comprehending the structure of these roots, a key found in the interchangeability of roots within a given polynomial and their intricate relationships with coefficients. The realization of this endeavor necessitated diligent work with Mathematica and, at times, a battle to persuade the software to unveil the secrets we sought. [10 - 13]

This article acknowledges the contributions of prior authors who have grappled with the enigmatic connection between symmetry and cubic equations, enriching our understanding of these mathematical phenomena and facilitating the production of this work.

2 Materials and Methods

All calculations concerning the search for a formula for the roots of a cubic equa-tion were performed in the mathematical software Wolfram Mathematica [14]. Wolf-ram Mathematica is presented as "the world's definitive system for modern technical computing". Mainly elements from Wolfram Language & System Documentation Center, specifically Symbolic & Numeric Computation were used.

As for the theoretical knowledge used in the article, these are three notorious statements from theory of polynomials of several indeterminates [15] (terms used in the paper are available at [16, 17]; elementary and basic symmetric polynomials mean the same thing - here the mathematical terminology is inconsistent.

- A polynomial $f(x_1,\dots,x_n)$ is called symmetric if $f(x_1,\dots,x_n) = f(x_{i_1},\dots,x_{i_n})$ for every permutation $(i_1,\dots,i_n)$ of the set $\{1,\dots,n\}$.
- To every symmetric polynomial $f(x_1,\dots,x_n)$, i.e. $f(x_1,\dots,x_n) = f(x_{i_1},\dots,x_{i_n})$ for every permutation $(i_1,\dots,i_n)$ of the set $\{1,\dots,n\}$, there exists a unique polynomial $g(u_1,\dots,u_n)$ such that $f(x_1,\dots,x_n) = g(\sigma_1(x_1,\dots,x_n),\dots,\sigma_n(x_1,\dots,x_n))$. The polynomial $g(\sigma_1(x_1,\dots,x_n),\dots,\sigma_n(x_1,\dots,x_n))$ is a polynomial of n uncertains where the uncertains are the basic symmetric polynomials $\sigma_1(x_1,\dots,x_n),\dots,\sigma_n(x_1,\dots,x_n)$.
- Let $x_1,\dots,x_n$ be roots of the algebraic equation $x^n + a_1x^{n-1} + \cdots a_{n-1}x + a_n = 0$. Then

$$-a_1 = x_1 + \cdots + x_n = \sigma_1(x_1,\dots,x_n)$$
$$a_2 = x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots + x_2x_n + \cdots + x_{n-1}x_n = \sigma_2(x_1,\dots,x_n)$$
$$\cdots\cdots\cdots$$
$$(-1)^n a_n = x_1x_2 \dots x_n = \sigma_n(x_1,\dots,x_n).$$

3 Results

3.1 Cardano's formula for solving cubic equations

We shall deal with cubic algebraic equations over the field of complex numbers. These are equations of the form.

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \qquad (1)$$

Where a_3, a_2, a_1, a_0 are complex and $a_3 \neq 0$.. Each such equation can be adjusted to its equivalent normed form.

$$x^3 + ax^2 + bx + c = 0 \qquad (2)$$

If the roots of the algebraic equation can be obtained from its coefficients in radicals then we say the formula for solving the equation exists. It is well known that while the formula for solving cubic equations (and quadratic and quartic) exists, no formula for algebraic equations of degree five or higher could be in general found. As stated above we shall deal only with a formula for solving cubic equations in this paper. This formula, called Cardano's formula, could be gotten very easily from Wolfram Mathematica using, for example, the command "Solve":

$$\text{Solve}[x\verb|^|3 + ax\verb|^|2 + bx + c == 0, x]$$

$$\{\{x \to -\frac{a}{3} - \frac{2^{1/3}(-a^2+3b)}{3\left(-2a^3+9ab-27c+3\sqrt{3}\sqrt{-a^2b^2+4b^3+4a^3c-18abc+27c^2}\right)^{1/3}} + \frac{\left(-2a^3+9ab-27c+3\sqrt{3}\sqrt{-a^2b^2+4b^3+4a^3c-18abc+27c^2}\right)^{1/3}}{32^{1/3}}\},$$

$$\left\{x \to -\frac{a}{3} + \frac{(1+i\sqrt{3})(-a^2+3b)}{32^{2/3}\left(-2a^3+9ab-27c+3\sqrt{3}\sqrt{-a^2b^2+4b^3+4a^3c-18abc+27c^2}\right)^{1/3}} - \frac{(1-i\sqrt{3})\left(-2a^3+9ab-27c+3\sqrt{3}\sqrt{-a^2b^2+4b^3+4a^3c-18abc+27c^2}\right)^{1/3}}{62^{1/3}}\right\}, \quad (3)$$

$$\{x \to -\frac{a}{3} + \frac{(1-i\sqrt{3})(-a^2+3b)}{32^{2/3}(-2a^3+9ab-27c+3\sqrt{3}\sqrt{-a^2b^2+4b^3+4a^3c-18abc+27c^2})^{1/3}} - \frac{(1+i\sqrt{3})(-2a^3+9ab-27c+3\sqrt{3}\sqrt{-a^2b^2+4b^3+4a^3c-18abc+27c^2})^{1/3}}{62^{1/3}}\}\}$$

Unfortunately, the formula turned out to be not very practical for manual counting; in the 16th century, when Gerolamo Cardano published this formula, there were no other methods of calculations - no computers :), of course. For example, let us solve the equation $x^3 + 3x - 4 = 0$ using the above Cardano's formula. In this case $a = 0,\ b = 3,\ c = -4.$ To substitute these values into this formula is laborious and unusable, so we will use a simplified Cardano's formula derived for the equation $x^3 + px + q = 0$ as stated in every textbook of algebra:

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} \quad (4)$$

This formula looks much better. So, by substituting $p = 3,\ q = -4$ to this easier formula we obtain.

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} \quad (5)$$

A person without any computer substituting in the equation for x the number 1 easily verifies that 1 is the root of this equation.

$$1 = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} \quad (6)$$

Or not? A person with a computer uses e.g. the Wolfram Mathematica to solve this equation (he is not interested in any formulas) and finds (the command Solve[x^3+3 x-4==0,x,Complexes]) solutions as follows:

$$1, \tfrac{1}{2}\left(-1 - i\sqrt{15}\right), \tfrac{1}{2}\left(-1 + i\sqrt{15}\right) \quad (7)$$

Further, he can easily check that this equation has one real solution (left hand side of the equation is monotone and negative respectively positive for negative respectively positive numbers) and that the only real solution really equals 1 (again he puts x=1 in the equation $x^3 + 3x - 4 = 0$)). Therefore he used the following commnads in Mathematica N[Surd[2+5^(1/2),3]+Surd[2-5^(1/2),3]] for finding the identity $1 = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.

A man who used Cardano's formula and has computer tries to find if $1 = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$. Therefore he writes in Mathematica (he has no idea about default setting for complex numbers in algebraic equations solving):

N[(2+5^(1/2))^(1/3)+(2-5^(1/2))^(1/3)] (8)

And he finds result 1.92705 + 0.535233 i. We can imagine that it will confuse him a bit ...

Nevertheless, the Cardano's formula has a theoretical and historical importance because searching for solutions to cubic equations contributed to creating the theory of complex numbers. It is also interesting that such a formula exists at all! Why? It will be clear from the following considerations.

3.2 Vieta's formulas

Let us denote the roots of the polynomial function $f(x) = x^3 + ax^2 + bx + c$ by x_1, x_2, x_3. Then the decomposition of the polynomial $f(x)$into root factors is $f(x) = (x - x_1)(x - x_2)(x - x_3)$. If we multiply the root factors and compare the appropriate forms, we obtain in Wolfram Mathematica.

$$x^3 + ax^2 + bx + c = (x - x_1)(x - x_2)(x - x_3)$$
$$x^3 + ax^2 + bx + c = x^3 - x^2x_1 - x^2x_2 + xx_1x_2 - x^2x_3 + xx_1x_3 + xx_2x_3 - x_1x_2x_3 \quad (9)$$

From this we get $a = -x_1 - x_2 - x_3$, $b = x_1x_2 + x_1x_3 + x_2x_3$ and $c = -x_1x_2x_3$.

Let $\sigma_i\,(x_1, \ldots, x_n)$ denote a basic symmetric polynomial. Then we have got so called Vieta's formulas [5]:

$$-a = x_1 + x_2 + x_3 = \sigma_1\,(x_1, x_2, x_3)$$
$$b = x_1x_2 + x_1x_3 + x_2x_3 = \sigma_2\,(x_1, x_2, x_3) \quad (10)$$
$$-c = x_1x_2x_3 = \sigma_3\,(x_1, x_2, x_3)$$

These basic symmetric polynomials $\sigma_1\,(x_1, x_2, x_3), \sigma_2\,(x_1, x_2, x_3), \sigma_3\,(x_1, x_2, x_3)$ are the key factors when revealing a formula for roots of not only this cubic equation, but also of a general algebraic equation of order n (then we use basic symmetric polynomials $\sigma_1\,(x_1, \ldots, x_n), \ldots, \sigma_n\,(x_1, \ldots, x_n)$); recall for $n > 4$ none formula exists!

Namely, the existence of a formula for roots of a cubic equation means that every root of a cubic equation should be expressed in radicals. However, we have seen that all coefficients *a, b, c* are symmetric polynomials of the equation's roots x_1, x_2, x_3. Because any operation with symmetric polynomials gives symmetric polynomials again, it seems to be impossible to express something as asymmetric as the root $x_1 = x_1(x_1, x_2, x_3)$ by means of symmetric polynomials. Yet it goes, as will be seen from the calculations in Wolfram Mathematica. The reasoning is as follows.

Let us start with the polynomial.

$$p_1 = x_1 + \frac{1}{2}(-1 + i\,\sqrt{3}\,)\,x_2 + \left[\frac{1}{2}(-1 + i\,\sqrt{3}\,)\right]^2 x_3 \quad (11)$$

Clearly, this polynomial is not symmetric. On the other hand, the third power of p_1 appears (see below) to be symmetric. In addition, because of the theorem saying that to every symmetric polynomial there exists a unique polynomial where the uncertains (variables) are the basic symmetric polynomials, one can express the third power of this polynomial in radicals. Moreover, by adding to this polynomial another two polynomials.

$$p_2 = x_1 + [\frac{1}{2}(-1 + i\,\sqrt{3}\,)]^2\,x_2 + \frac{1}{2}(-1 + i\,\sqrt{3}\,)\,x_3 \quad (12)$$

$$p_3 = x_1 + x_2 + x_3 \quad (13)$$

We get $3x_1$,, i. e. (divided by 3) we obtain the first of three roots of our cubic equation.

$$p_1 + p_2 + p_3 = x_1 - \frac{x_2}{2} + \frac{1}{2}i\sqrt{3}x_2 - \frac{x_3}{2} - \frac{1}{2}i\sqrt{3}x_3 + x_1 - \frac{x_2}{2} - \frac{1}{2}i\sqrt{3}x_2 - \frac{x_3}{2} + \frac{1}{2}i\sqrt{3}x_3 + x_1 + x_2 + x_3 = 3x_1 \quad (14)$$

3.3 Reducing to basic symmetric polynomials

Let us start with equation:

$${p_1}^3 = \left(x_1 + \frac{1}{2}\left(-1 + i\sqrt{3}\,\right)x_2 + \frac{1}{4}\left(-1 + i\sqrt{3}\,\right)^2 x_3\right)^3 \quad (15)$$

There is the built-in symbol in the Wolfram Language & System called S"ymmetricReduction [*f*] " which reduces the *f* polynomial to the pair of polynomials {*polynomial of basic symmetric polynomials,remainder*}. By using this command we get:

$$\left(x_1 + \frac{1}{2}\left(-1 + i\sqrt{3}\,\right)x_2 + \frac{1}{4}\left(-1 + i\sqrt{3}\,\right)^2 x_3\right)^3 =$$
$$\left\{\sigma_1^3 + \left(-\frac{9}{2} + \frac{3i\sqrt{3}}{2}\right)\sigma_1\sigma_2 + 3\left(\frac{9}{2} - \frac{3i\sqrt{3}}{2}\right)\sigma_3, -3i\sqrt{3}x_1x_2^2 - 3i\sqrt{3}x_1^2x_3 - 3i\sqrt{3}x_2x_3^2\right\} \quad (16)$$

However, this reduction does not help us because the reminder is not the null poly-nomial. Therefore, we will proceed as follows:

$$\left(x_1 + \frac{1}{2}(-1 + i\sqrt{3})x_2 + \frac{1}{4}(-1 + i\sqrt{3})^2 x_3\right)^3 \\ = x_1^3 - \frac{3}{2}x_1^2x_2 + \frac{3}{2}i\sqrt{3}x_1^2x_2 - \frac{3}{2}x_1x_2^2 - \frac{3}{2}i\sqrt{3}x_1x_2^2 + x_2^3 - \frac{3}{2}x_1^2x_3 - \frac{3}{2}i\sqrt{3}x_1^2x_3 + 6x_1x_2x_3 \\ - \frac{3}{2}x_2^2x_3 + \frac{3}{2}i\sqrt{3}x_2^2x_3 - \frac{3}{2}x_1x_3^2 + \frac{3}{2}i\sqrt{3}x_1x_3^2 - \frac{3}{2}x_2x_3^2 - \frac{3}{2}i\sqrt{3}x_2x_3^2 + x_3^3 \tag{17}$$

Now we start with the real part.

$$x_1^3 - \frac{3}{2}x_1^2x_2 - \frac{3}{2}x_1x_2^2 + x_2^3 - \frac{3}{2}x_1^2x_3 + 6x_1x_2x_3 - \frac{3}{2}x_2^2x_3 - \frac{3}{2}x_1x_3^2 - \frac{3}{2}x_2x_3^2 + x_3^3 \tag{18}$$

And by using the built-in symbol "SymmetricReduction [*f*]" we get:

$$\left\{\sigma_1^3 - \frac{9\sigma_1\sigma_2}{2} + \frac{27\sigma_3}{2}, 0\right\} \tag{19}$$

As for the imaginary part.

$$\frac{3}{2}i\sqrt{3}x_1^2x_2 - \frac{3}{2}i\sqrt{3}x_1x_2^2 - \frac{3}{2}i\sqrt{3}x_1^2x_3 + \frac{3}{2}i\sqrt{3}x_2^2x_3 + \frac{3}{2}i\sqrt{3}x_1x_3^2 - \frac{3}{2}i\sqrt{3}x_2x_3^2 \tag{20}$$

The situation is a bit more complicated. We have to use the built-in symbol SymmetricReduction [*f*] for the second power of this polynomial, i.e. for:

$$(\frac{3}{2}i\sqrt{3}x_1^2x_2 - \frac{3}{2}i\sqrt{3}x_1x_2^2 - \frac{3}{2}i\sqrt{3}x_1^2x_3 + \frac{3}{2}i\sqrt{3}x_2^2x_3 + \frac{3}{2}i\sqrt{3}x_1x_3^2 - \frac{3}{2}i\sqrt{3}x_2x_3^2)^2 \tag{21}$$

Then we get:

$$\left\{-\frac{27}{4}\sigma_1^2\sigma_2^2 + 27\sigma_2^3 + 27\sigma_1^3\sigma_3 - \frac{243}{2}\sigma_1\sigma_2\sigma_3 + \frac{729\sigma_3^2}{4}, 0\right\} \tag{22}$$

The square root of the polynomial where the variables are the basic symmetric polynomials looks like this:

$$\sqrt{-\frac{27}{4}\sigma_1^2\sigma_2^2 + 27\sigma_2^3 + 27\sigma_1^3\sigma_3 - \frac{243}{2}\sigma_1\sigma_2\sigma_3 + \frac{729\sigma_3^2}{4}} \tag{23}$$

Hence, we have finally obtained.

$$x_1 + \frac{1}{2}(-1 + i\sqrt{3})\, x_2 + [\frac{1}{2}(-1 + i\sqrt{3})]^2\, x_3 = \\ \left(\sigma_1^3 - \frac{9\sigma_1\sigma_2}{2} + \frac{27\sigma_3}{2} + \frac{3}{2}\sqrt{3}\sqrt{-\sigma_1^2\sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3\sigma_3 - 18\sigma_1\sigma_2\sigma_3 + 27\sigma_3^2}\right)^{1/3} \tag{24}$$

We will do the same for the second polynomial $p_2 = x_1 + [\frac{1}{2}(-1 + i\sqrt{3})]^2\, x_2 + \frac{1}{2}(-1 + i\sqrt{3})\, x_3$.

$$x_1 + \frac{1}{4}(-1 + i\sqrt{3})^2 x_2 + \frac{1}{2}(-1 + i\sqrt{3})\, x_3 = \\ \left(\sigma_1^3 - \frac{9\sigma_1\sigma_2}{2} + \frac{27\sigma_3}{2} + \frac{3}{2}\sqrt{3}\sqrt{\sigma_1^2\sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3\sigma_3 - 18\sigma_1\sigma_2\sigma_3 + 27\sigma_3^2}\right)^{1/3} \tag{25}$$

Now by adding the polynomials $p_1 + p_2 + p_3$ and dividing by 3 we get the formula for the root x_1:

$$x_1 = \frac{1}{3}([x_1 + \frac{1}{2}\left(-1 + i\sqrt{3}\right) x_2 + \left[\frac{1}{2}\left(-1 + i\sqrt{3}\right)\right]^2 x_3] + [x_1 + \left[\frac{1}{2}\left(-1 + i\sqrt{3}\right)\right]^2 x_2 \\ + \frac{1}{2}\left(-1 + i\sqrt{3}\right) x_3] + [x_1 + x_2 + x_3]) = \\ \frac{1}{6}\left(2\sigma_1 + 2^{\frac{2}{3}}\left(\left(2\sigma_1^3 - 9\sigma_1\sigma_2 + 27\sigma_3 + 3\sqrt{3}\sqrt{-\sigma_1^2\sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3\sigma_3 - 18\sigma_1\sigma_2\sigma_3 + 27\sigma_3^2}\right)^{\frac{1}{3}} \right.\right. \\ \left.\left. + \left(2\sigma_1^3 - 9\sigma_1\sigma_2 + 27\sigma_3 + 3\sqrt{3}\sqrt{\sigma_1^2\sigma_2^2 + 4\sigma_2^3 + 4\sigma_1^3\sigma_3 - 18\sigma_1\sigma_2\sigma_3 + 27\sigma_3^2}\right)^{\frac{1}{3}}\right)\right) \tag{26}$$

We conclude we have found the formula for the first root x_1 of the cubic equation then.

Similarly, as for the root x_2 we start with the sum of these three polynomials.

$$[\frac{1}{2}(-1+i\sqrt{3})\,x_1+x_2+\frac{1}{4}(-1+i\sqrt{3})^2x_3]+[\frac{1}{4}(-1+i\sqrt{3})^2x_1+x_2+\frac{1}{2}(-1+i\sqrt{3})\,x_3]+[x_1+x_2+x_3]$$
$$=\frac{x_1}{2}+\frac{1}{2}i\sqrt{3}x_1+x_2-\frac{x_3}{2}-\frac{1}{2}i\sqrt{3}x_3-\frac{x_1}{2}-\frac{1}{2}i\sqrt{3}x_1+x_2-\frac{x_3}{2}+\frac{1}{2}i\sqrt{3}x_3+x_1+x_2+x_3=3x_2 \quad (27)$$

The next procedure for finding a formula for x_2 could be the same like for the root x_1; we will not perform this calculation therefore.

Finally, for the root x_3 the sum of corresponding three polynomials looks like this:

$$[\frac{1}{2}(-1+i\sqrt{3})\,x_1+\frac{1}{4}(-1+i\sqrt{3})^2x_2+x_3]+[\frac{1}{4}(-1+i\sqrt{3})^2x_1+\frac{1}{2}(-1+i\sqrt{3})\,x_2+x_3]+[x_1+x_2+x_3]$$
$$=-\frac{x_1}{2}+\frac{1}{2}i\sqrt{3}x_1-\frac{x_2}{2}-\frac{1}{2}i\sqrt{3}x_2+x_3-\frac{x_1}{2}-\frac{1}{2}i\sqrt{3}x_1-\frac{x_2}{2}+\frac{1}{2}i\sqrt{3}x_2+x_3+x_1+x_2+x_3=3x_3 \quad (28)$$

And we will continue in the same way as in the case of roots x_1, x_2.

4 Discussion

The article tries to demonstrate that the laborious and tedious calculations needed to reveal a variety of nice mathematical relationships could be successfully replaced by mathematical software, specifically by a computer algebra system. The contribution of the article is then a concrete example of the fact that without a proper understanding of the nature of the problem, computer calculations do not lead to reasonable conclusions. As mentioned in the article, it was necessary to divide certain terms during calculations to gain the needed polynomial of basic symmetric polynomials; this part can be considered, in the opinion of the authors, as the original. Note further that this approach to deriving a formula for the roots of a cubic equation makes it possible to show that no such formulas for equations of order5 and higher can exist. As also mentioned in the article, it is even surprising that formulas for the roots of at least some algebraic equations exist at all. Be-cause the arithmetic operations with symmetric polynomials (and each coefficient of an algebraic equation is a symmetric polynomial on its roots - see Vieta's formulas) again form symmetric polynomials. And, in contrast, each root is obviously an asymmetric polynomial.

5 Conclusions

In the relentless pursuit of optimization in the realm of engineering, this article has delved into the intricate relationship between cubic equations and the practical world of production technologies. Through this exploration, we've unearthed profound connections that underpin the vital role of mathematical symmetry in the field of engineering. The journey has unveiled several key insights and takeaways:

Practical Significance of Cubic Equations: Cubic equations have proven to be indispensable tools in understanding and enhancing real-world applications within engineering and manufacturing technology. These equations often arise when analyzing mechanical systems, electric circuits, robotics, and other complex systems. The insights gained from these mathematical models have a direct impact on optimizing designs, improving system performance, and making informed decisions.

The Connection Between Symmetry and Engineering: The significance of symmetry, a mathematical concept with both theoretical and practical implications, has been under-scored in the study of cubic equations. Vieta's formulas from the 16th century and the subsequent work of Abel and Galois in the 19th century have highlighted the role of symmetry in understanding algebraic equations, particularly when they extend beyond the fourth degree.

Challenges and Insights in Finding Roots: The study has unveiled the complexity of finding roots of cubic equations, especially when expressed in radicals. While the general cubic formula may not always provide straightforward solutions, it has shed light on the interchangeability of roots within a given polynomial and their intricate relationships with coefficients. This understanding holds the key to deciphering the structure of these roots.

Mathematics as a Foundation for Engineering: The critical role of mathematics in engineering and manufacturing technology cannot be understated. It serves as the foundation for modeling, analyzing, and optimizing various systems and processes, contributing to advancements and innovations in the field.

While this article has not presented a definitive formula for cubic equation roots, it has illuminated the path toward comprehending their structure and the intricate relationships that exist within these equations. Furthermore, it has emphasized the practical importance of cubic equations and the role of mathematics in optimizing manufacturing technology.

In the ever-evolving landscape of engineering and manufacturing technology, the journey to harness the power of cubic equations and mathematical symmetry continues. The insights gained here serve as a steppingstone for future research and application, in-spiring engineers and mathematicians to unlock the full potential of these mathematical tools in advancing technology and optimizing manufacturing processes.

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