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Nonlinear Stochastic Dynamics Analysis of Vehicle Bodies Based on the Direct Probability Density Integral Method

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Manufacturing inaccuracies in vehicle suspension systems inevitably lead to uncertainties in the parameters of their structural components. Simultaneously, the road excitation impacting nonlinear vehicle systems exhibits pronounced randomness and time-variant characteristics. Consequently, it is crucial to conduct a stochastic dynamics analysis on nonlinear suspension systems, taking into account these uncertain factors. In this paper, a seven-degree-of-freedom (7-DOF) nonlinear suspension system dynamics model has been established. The stochastic process of road irregularities is simulated using the harmonic superposition method. Moreover, based on the direct probability density integral method, the stochastic dynamic equations of the nonlinear suspension system and their corresponding solution strategies have been developed and explored. Through MATLAB, the time-varying probability density function of the vibration response for a nonlinear vehicle suspension system was calculated under the combined effects of stochastic road irregularity excitation and random coupling of system structural parameters. Additionally, analyses were conducted on how different coefficients of variation and the intensity of nonlinearity in the suspension system influence the probability density of the output body displacement of the nonlinear vehicle suspension system. The research outcomes demonstrate that the direct probability density integral method offers superior efficiency and accuracy when computing nonlinear vehicle suspension systems. Furthermore, altering the coefficients of variation for various system parameters reveals that as these coefficients increase, the disparity in the probability density of body displacement becomes more pronounced, leading to more intense vehicle vibrations. Under soft nonlinear conditions with lower suspension spring stiffness, the probability density function of body displacement shifts slightly to the right with minimal change. However, under strong nonlinear conditions, body displacement significantly increases, resulting in diminished vibration isolation capabilities of the suspension system. This leads to severe jolts and a noticeable decline in ride comfort during vehicle operation.

Keyword: Direct probability density integral method, The road is not smooth, Harmonic superposition method, Nonlinear vehicle suspension system

1 Introduction

When a vehicle travels at high speeds over uneven surfaces, it encounters excitations from the road's irregularities, leading to vibrations and noise. This stochastic vibration, transmitted through the wheels to the vehicle body, can result in intense shocks that accelerate wear between components, reducing their service life and impacting the vehicle's handling stability. For the driver, continuous jolts can cause fatigue and distract attention, potentially leading to traffic accidents and causing loss of life and property [1]. Currently, many vehicles employ hydraulic dampers that exhibit certain levels of nonlinearity. When a vehicle is traveling at high speeds, a higher spring stiffness is required to ensure handling stability; conversely, at low speeds, a lower spring stiffness is needed to maintain ride comfort, indicating a nonlinear characteristic of the springs. Additionally, the nonlinearity is also present in vehicle tires due to the effects of inflation pressure, aging, wear, and temperature [2]. Moreover, during the production process of vehicles, due to the negligence of staff and measurement and precision errors of processing machinery, it is inevitable that there will be uncertainties in parameters such as sprung mass, unsprung mass, suspension stiffness, suspension damping, and tire stiffness [3-4]. In addition, due to the complexity of road conditions, vehicles will receive random excitations from different road surfaces during driving. Therefore, studying the nonlinear and stochastic coupling dynamic response and control mechanism of vehicles is important for comfort.

The stochastic vibration analysis of nonlinear vehicle suspension systems mainly involves solving the statistical regularities of the system's output random response. It analyzes the influence of the system's external environment and its own parameters on the safety and comfort of vehicle driving, ensuring more stability and safety when the vehicle is driving at high speed. Niu and Wu [5] established a nonlinear model of single-degree-of-freedom suspension using a power function polynomial and studied the dynamic response of the system under bounded noise and C-level road excitation, respectively. Based on the pseudo-excitation method and equivalent linearization technology, Hua et al. [6] constructed a stable iterative scheme and used this method to analyze the power spectral density of vehicle responses under different speeds and different nonlinear hysteretic springs. Jia et al. [7] established a five-degree-of-freedom nonlinear vehicle model, considering the randomness of structural physical parameters in space and the randomness of acting loads in time. The response of the nonlinear system under random process excitation was analyzed by solving the Lyapunov equation. Yi et al. [8] established the dynamic equation of vehicle random vibration, considering the randomness of vehicle structural parameters. The system's phase response variation was used to reflect the influence of compound randomness on the vibration response of the vehicle structure. However, it only analyzed the frequency domain response of the linear system, while the vibration of actual vehicles is mostly a nonlinear vibration problem. The above-mentioned studies rarely consider the coupled vibration of the vehicle body under the dual randomness of vehicle nonlinearity, road roughness, and system structure randomness.

Currently, there are two methods for analyzing the stochastic dynamics of nonlinear vehicle suspension systems: frequency domain and time domain. The frequency domain method mainly analyzes the power spectral density function and is primarily suitable for the stochastic dynamics analysis of linear systems [9]. Different methods have been developed to obtain probabilistic information for the stochastic response of various structural systems. For instance, the Fokker-Planck-Kolmogorov (FPK) [10] equation method, the equivalent linearization method [11], the stochastic averaging method [12], and the Monte Carlo method [13-14]. The response of a nonlinear system excited by a Gaussian white noise process is a Markov process, and its probability density function needs to be obtained by solving the Fokker-Planck-Kolmogorov (FPK) equation. However, solving partial differential equations is challenging, and the FPK equation, which involves coupling between the state space and the physical system, is difficult to apply to nonlinear systems with a large number of degrees of freedom [15]. Additionally, Monte Carlo Simulation (MCS) is a general method for nonlinear systems with random parameters and/or random excitations [16]. However, due to its high computational cost, it is typically used to verify the accuracy of problems. Lin et al. [17] established and developed an efficient and precise virtual excitation method for random vibration analysis. By constructing virtual input excitations, the response power spectrum and statistical information can be quickly and easily obtained. Li and Chen [18] derived the Generalized Density Evolution Equation (GDEE). By solving the GDEE partial differential equation, the probability density evolution result of the target response can be obtained, which improves efficiency compared to MCS. However, during the calculation process, the discretized time step and space step need to satisfy the Courant-Friedrichs-Lewy (CFL) condition. Chen and Yang [19] proposed the Direct Probability Density Integration Method (DPIM) based on the principle of probability conservation. By introducing the Dirac function, they derived the Probability Density Integral Equation (PDIE) for static (dynamic) systems, which characterizes the explicit relationship between input and output probability densities. DPIM not only breaks through the CFL limitation but also greatly improves the computational efficiency for random dynamic analysis of large nonlinear dynamic (static) systems by using GF deviation point selection and smoothing technology of the Dirac delta function to directly solve the PDIE.

This paper establishes a nonlinear model of a complete vehicle with seven degrees of freedom. Using the harmonic superposition method, time-domain samples of random road roughness are constructed. Additionally, dynamic equations for a vehicle system with random parameters under random excitations are formulated. The Direct Probability Density Integration Method (DPIM) is employed to analyze the random vibration of the nonlinear vehicle system influenced by both road roughness and structural parameters. The applicability of DPIM in the analysis of nonlinear and stochastic coupled vibrations is verified through comparison with Monte Carlo Simulation (MCS). The probability evolution process of the system's random time-varying response is obtained, and the effects of different coefficients of variation and nonlinear intensities on the nonlinear random response of the vehicle body are analyzed.

2 Dynamic model of seven-degree-of-freedom vehicle suspension with uncertain parameters

Vehicle structures encompass numerous nonlinear elements, and the wear and aging of automotive parts during use further increase these nonlinear factors. Therefore, it is indispensable to study nonlinear vehicle systems. This paper only considers the nonlinear stiffness characteristics of vehicle suspension springs, and the relationship between force and displacement can be approximately expressed as [20]:

$$F_s = kx + \varepsilon kx^3 \quad (1)$$

Where:

F_S …The spring force;

k…The stiffness of the spring;

x…The stretched length of the spring;

ε…The nonlinearity of the spring stiffness.

When $\varepsilon = 0$, the suspension system can be regarded as a linear system.

The vehicle suspension model is simplified into a seven-degree-of-freedom spring oscillation model, as shown in Figure 1. Assuming the vehicle body is a rigid body, when the vehicle moves in a straight line at a uniform speed on a horizontal road, the body will move in three directions: vertical, roll, and pitch. Simultaneously, the four wheels of the vehicle will move in the vertical direction. Some of the parameters are listed in Table 1.

***Tab. 1** Structural parameters of the seven-degree-of-freedom model*

Parameters	Notation	Unit
m_s	Sprung mass	kg
z_s	Vertical displacement of the center of mass	m
θ	Body Angle around x axis	rad
φ	Body Angle around the y axis	rad
I_x	Moment of inertia about the X-axis	kg·m²
I_y	Moment of inertia about the Y-axis	kg·m²
m_1, m_2, m_3, m_4	Unsprung mass	kg
z_1, z_2, z_3, z_4	Vertical displacement of unsprung mass	m
$z_{s1}, z_{s2}, z_{s3}, z_{s4}$	Body vertical displacement	m
k_1, k_2, k_3, k_4	Suspension spring stiffness	N/m
c_1, c_2, c_3, c_4	Damping coefficient of suspension shock absorber	N/(m/s)
$k_{t1}, k_{t2}, k_{t3}, k_{t4}$	Tire dynamic stiffness	N/m
l_1	The distance between the center of mass and the front axle	m
l_2	Distance from the center of mass to the rear axis	m
l_3	Half the distance from the front axle of the vehicle	m
l_4	Half the distance from the rear axle of the vehicle	m
q	Pavement excitation input	m

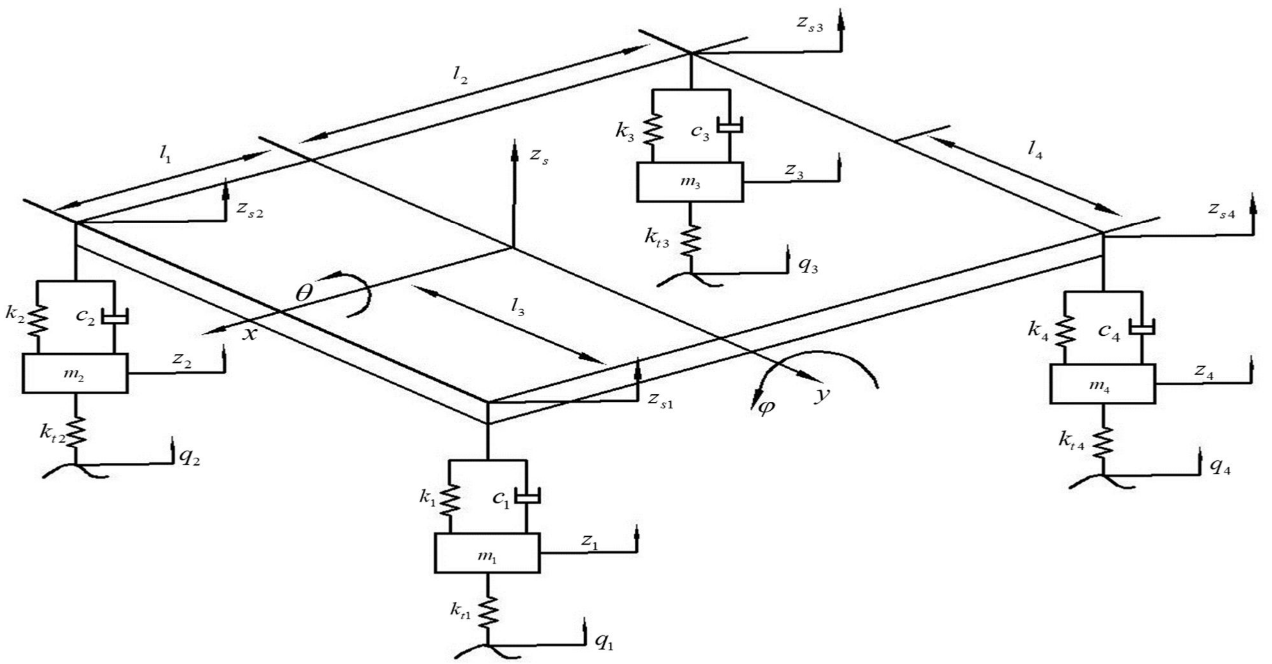

***Fig. 1** Nonlinear suspension model of a seven-degree-of-freedom vehicle*

Considering that the damping of tires is very small, it is neglected in this paper. Due to the uncertainty of the mass, damping, and stiffness of the suspension system, taking the stiffness of the suspension spring as an example, the spring stiffness is regarded as a random parameter that follows a normal distribution, denoted as $k \sim N(\mu_k, \sigma_k)$, where μ_k represents the mean value of the spring stiffness, and σ_k denotes its standard deviation.

The nonlinear dynamic equation of this system can be expressed as:

$$M\ddot{X} + C\dot{X} + KX + G(X, \ddot{X}) = F(t) \tag{2}$$

Where:

$$M = \begin{bmatrix} m_s & & & & & & \\ & I_y & & & & & \\ & & I_x & & & & \\ & & & m_1 & & & \\ & & & & m_2 & & \\ & & & & & m_3 & \\ & & & & & & m_4 \end{bmatrix}; \quad G = \varepsilon \begin{bmatrix} k_1 z_1^3 + k_2 z_2^3 + k_3 z_3^3 + k_4 z_4^3 \\ -k_1 z_1^3 - k_2 z_2^3 + k_3 z_3^3 + k_4 z_4^3 \\ -k_1 z_1^3 + k_2 z_2^3 + k_3 z_3^3 - k_4 z_4^3 \\ -k_1 z_1^3 \\ -k_2 z_2^3 \\ -k_3 z_3^3 \\ -k_4 z_4^3 \end{bmatrix};$$

$$K = \begin{bmatrix} k_1+k_2+k_3+k_4 & -l_1k_1-l_1k_2+l_2k_3+l_2k_4 & -l_3k_1+l_3k_2+l_4k_3-l_4k_4 & -k_1 & -k_2 & -k_3 & -k_4 \\ -l_1k_1-l_1k_2+l_2k_3+l_2k_4 & l_1^2k_1+l_1^2k_2+l_2^2k_3+l_2^2k_4 & l_1l_3k_1-l_1l_3k_2+l_2l_4k_3-l_2l_4k_4 & l_1k_1 & l_1k_2 & -l_2k_3 & -l_2k_4 \\ -l_3k_1+l_3k_2+l_4k_3-l_4k_4 & l_1l_3k_1-l_1l_3k_2+l_2l_4k_3-l_2l_4k_4 & l_3^2k_1+l_3^2k_2+l_4^2k_3+l_2^2k_4 & l_3k_1 & -l_3k_2 & -l_4k_3 & l_4k_4 \\ -k_1 & l_1k_1 & l_3k_1 & k_1+k_{t1} & 0 & 0 & 0 \\ -k_2 & l_1k_2 & -l_3k_2 & 0 & k_2+k_{t2} & 0 & 0 \\ -k_3 & -l_2k_3 & -l_4k_3 & 0 & 0 & k_3+k_{t3} & 0 \\ -k_4 & -l_2k_4 & l_4k_4 & 0 & 0 & 0 & k_4+k_{t4} \end{bmatrix};$$

$$C = \begin{bmatrix} c_1+c_2+c_3+c_4 & -l_1c_1-l_1c_2+l_2c_3+l_2c_4 & -l_3c_1+l_3c_2+l_4c_3-l_4c_4 & -c_1 & -c_2 & -c_3 & -c_4 \\ -l_1c_1-l_1c_2+l_2c_3+l_2c_4 & l_1^2c_1+l_1^2c_2+l_2^2c_3+l_2^2c_4 & l_1l_3c_1-l_1l_3c_2+l_2l_4c_3-l_2l_4c_4 & l_1c_1 & l_1c_2 & -l_2c_3 & -l_2c_4 \\ -l_3c_1+l_3c_2+l_4c_3-l_4c_4 & l_1l_3c_1-l_1l_3c_2+l_2l_4k_3-l_2l_4k_4 & l_3^2c_1+l_3^2c_2+l_4^2c_3+l_2^2c_4 & l_3c_1 & -l_3c_2 & -l_4c_3 & l_4c_4 \\ -c_1 & l_1c_1 & l_3c_1 & c_1 & 0 & 0 & 0 \\ -c_2 & l_1c_2 & -l_3c_2 & 0 & c_2 & 0 & 0 \\ -c_3 & -l_2c_3 & -l_4c_3 & 0 & 0 & c_3 & 0 \\ -c_4 & -l_2c_4 & l_4c_4 & 0 & 0 & 0 & c_4 \end{bmatrix};$$

$$X = \begin{bmatrix} x_s & \theta & \varphi & x_1 & x_2 & x_3 & x_4 \end{bmatrix}^{\mathrm{T}};$$

$$F(t) = \begin{bmatrix} 0 & 0 & 0 & k_{t1}q_1 & k_{t2}q_2 & k_{t3}q_3 & k_{t4}q_4 \end{bmatrix}^{\mathrm{T}};$$

$$z = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} x_{s1}-x_1 & x_{s2}-x_2 & x_{s3}-x_3 & x_{s4}-x_4 \end{bmatrix}^{\mathrm{T}}.$$

In this paper, the physical parameters of all structures in the vehicle system are regarded as normally distributed random variables. Therefore, the randomness of the system parameters leads to randomness in the system's mass matrix, stiffness matrix, damping matrix, and matrices G. Equation (2) represents the dynamic equation of a seven-degree-of-freedom nonlinear vehicle system under the dual randomness of random parameters and random excitations.

3 Time domain model of road roughness

According to the national standard, the power spectral density of road roughness is expressed as [21]:

$$G_d(n) = G_d(n_0)\left(\frac{n}{n_0}\right)^{-w} \tag{3}$$

Where:

n …Spatial frequency, with a unit of m^{-1};

n_0 …Reference spatial frequency, usually n0=0.1 m^{-1};

$G_d(n)$ …Road surface power spectral density, with a unit of m^2 / m^{-1};

$G_d(n_0)$ …Road roughness coefficient;

w …Frequency index, generally w =2.

When analyzing the dynamic performance of a vehicle during driving, the driving speed of the vehicle is an unavoidable factor. Therefore, it is necessary to convert spatial power spectral density $G_d(n)$ to temporal power spectral density $G_d(f)$. The relationship between spatial frequency and temporal frequency can be converted as follows:

$$f = un \tag{4}$$

Further, the formula for transforming the power spectral density of space frequency into that of time frequency can be derived as follows:

$$G_d(f) = \frac{1}{u} G_d(n) \tag{5}$$

The expression for temporal power spectral density is as follows:

$$G_d(f) = G_d(n_0) n_0^{\ 2} \frac{u}{f^2} \tag{6}$$

In this paper, the harmonic superposition method [22] is adopted to generate time-domain samples of random road roughness. The power spectral density $G_d(f)$ of the time frequency is expanded according to the average power spectrum, and the variance of road roughness is:

$$\sigma_z^{\ 2} = \int_{f_1}^{f_2} G_d(f) df \tag{7}$$

Divide the time-frequency domain into m small intervals, and then take the intermediate frequency f_{mid-i} of each small interval to calculate $G_d(f_{mid-i})$ for each interval to replace $G_d(f)$, then $\sigma_z^{\ 2}$ can be written as:

$$\sigma_z^{\ 2} \approx \sum_{i=1}^{m} G_d(f_{\min-i}) \cdot \Delta f \tag{8}$$

Then the sine wave function of the standard deviation between each cell can be expressed as:

$$\sqrt{2G_d(f_{\min-i}) \cdot \Delta f} \cdot \sin(2\pi f_{\mathrm{mi}\,d-i} t + \theta_i) \tag{9}$$

Then, by superimposing the sine wave functions of each small interval, we obtain the random displacement of the road surface in the time domain:

$$q(t) = \sum_{i=1}^{m} \sqrt{2G_d(f_{\min-i}) \cdot \Delta f} \cdot \sin(2\pi f_{\mathrm{mi}\,d-i} t + \theta_i) \tag{10}$$

Where:

Δf …The time frequency step size;

t …Time;

θ_i …Uniformly distributed random variable on [0, 2π].

4 Direct probability density integral solution

4.1 Probability density integral equation

In nature, there are certain physical quantities that maintain a constant value, such as conservation of mass and conservation of energy. In fact, the probability carried by random events is also conserved. In the process of state changes, if no other random factors are introduced and no random factors disappear, the system can be considered to maintain probability conservation.

Based on the principle of probability conservation, considering the random vector entering the nonlinear vehicle suspension system as $\boldsymbol{\theta}$ (which includes random parameters or loads), and the random vector of the system output as $\boldsymbol{X}$, the principle of probability conservation in the system manifests as follows:

$$\int_{\Omega_x} p_{\mathrm{X}}(\mathbf{x},t)\mathrm{d}\mathbf{x} = \int_{\Omega_{\Theta_0}} p_{\Theta}(\boldsymbol{\theta},t_0)\mathrm{d}\boldsymbol{\theta} \tag{11}$$

Where:

$\boldsymbol{\Theta}$ …The random input vector, which includes all random factors introduced into the nonlinear vehicle suspension at the initial time t_0;

$\mathbf{X}$ …The dynamic response vector output by the system;

$p_{\Theta}(\boldsymbol{\theta},t_0)$, $p_{\mathbf{X}}(\mathbf{x},t)$ …The probability density functions (PDF) of the input vector $\boldsymbol{\Theta}$ and output vector $\mathbf{X}$ at times t_0 and t, respectively;

Ω_{Θ} , $\Omega_{\mathbf{X}}$ …The sample spaces corresponding to the input and output random vectors, respectively.

The randomness of the road-vehicle suspension system is transmitted from the input random vector $\boldsymbol{\Theta}$ to the output vector $\mathbf{X}(t)$, and this transmission relationship can be represented by the mapping G:

$$G : \mathbf{X}(t) = g\left[\boldsymbol{\Theta}_s, I(\boldsymbol{\Theta}_f, t)\right] = g(\boldsymbol{\Theta}, t) \tag{12}$$

Where:

$\boldsymbol{\Theta} = [\boldsymbol{\Theta}_s, \boldsymbol{\Theta}_f]$ …Collects all the basic random variables of the system;

$\boldsymbol{\Theta}_s$ and $\boldsymbol{\Theta}_f$ …The random parameters of various structures in the vehicle system and the random vector of road excitation, respectively.

Based on probability theory, the probability density function $p_{\mathbf{X}}(\mathrm{x},t)$ of the output can be expressed in terms of the probability density functions $p_{\Theta}(\theta)$ of the input variables.

$$p_Y(\mathbf{x},t) = \left|\mathbf{J}_{g^{-1}}(\mathbf{x})\right| p_{\Theta}\left[\boldsymbol{\theta} = \boldsymbol{g}^{-1}(\mathbf{x},t)\right] = \frac{1}{\left|\mathbf{J}_{g}(\boldsymbol{\theta})\right|} p_{\Theta}\left[\boldsymbol{\theta} = \boldsymbol{g}^{-1}(\mathbf{x},t)\right] \quad (13)$$

Where:

$\left|\mathbf{J}_{g^{-1}}(\mathrm{x})\right| = \left|\frac{\partial[g^{-1}(\mathrm{x})]}{\partial \mathrm{x}}\right|$, $\left|\mathbf{J}_{g}(\theta)\right| = \left|\frac{\partial[g(\theta)]}{\partial \theta}\right|$ …The determinant of the Jacobian matrix.

In a general stochastic system, the mapping G may be explicit or implicit. In the explicit case, an analytical expression for the output response can be directly obtained through equations, and then the probability density function of the output response can be directly obtained through the probability density integral equation. However, the road-vehicle suspension system is a complex nonlinear system, and the mapping G is usually implicit, making it impossible to directly calculate the output response of the system. Therefore, its corresponding inverse function $g^{-1}(\cdot)$ and Jacobian determinant $\left|\mathrm{J}_g(\theta)\right|$ are difficult to determine. Based on the sifting property of the Dirac delta function, we use the Dirac delta function to solve the inverse function and Jacobian matrix J. At this point, the probability density function of the n-dimensional basic random vector $\boldsymbol{\Theta} = \left[\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, ..., \boldsymbol{\Theta}_n\right]$ can be rewritten as:

$$p_{\Theta}(\boldsymbol{\theta}) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} p_{\Theta}(\mathbf{s}) \delta(\mathrm{s} - \boldsymbol{\theta}) \mathrm{d}\mathbf{s} \quad (14)$$

Where:

$\delta(\mathbf{s} - \boldsymbol{\theta}) = \delta(s_1 - \theta_1) \cdots \delta(s_n - \theta_n)$;

$\mathrm{d}\mathbf{s} = ds_1 \cdots ds_n$;

$\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_n)$;

$\mathbf{s} = (s_1, s_2, \cdots, s_n)$;

$\boldsymbol{s}$, $\boldsymbol{\theta}$ …Symmetrical to each other in the random vector Θ.

Based on the Dirac delta function variable transformation formula and symmetry, as well as the equation G , the transformation of probability from the random input vector Θ to the random output response $\mathbf{X}$ can be expressed in the form of an integral. Therefore, the probability density function of the system's random response $\mathrm{X}(t)$ at time t is:

$$p_{\mathbf{X}}(\mathbf{x},t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\Theta}(\boldsymbol{\theta}) \delta\left[\mathbf{x} - \boldsymbol{g}(\boldsymbol{\theta},t)\right] \mathrm{d}\boldsymbol{\theta} \quad (15)$$

When we want to obtain the random probability density function of a certain component of the nonlinear vehicle suspension system, we can integrate both sides of the above equation to obtain the marginal PDF of the random response $x_l(t)$:

$$p_{Y_l}(x_l,t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\Theta}(\boldsymbol{\theta}) \delta\left[x_l - g_l(\boldsymbol{\theta},t)\right] \mathrm{d}\boldsymbol{\theta} \quad (16)$$

4.2 PDIE is solved by direct probability density integral

Due to the difficulty of obtaining the response quantity of the nonlinear vehicle suspension system through analytical solutions, an efficient and accurate method is needed to solve the PDIE (Probability Density Integral Equation) in the equation. Chen et al. [23] proposed a new method to solve the probability density function of random output response, which is called Direct Probability Integral Method (DPIM). It consists of two key techniques: (1) Probability space subdivision, which utilizes the GF discrepancy point selection technique in the probability density evolution method to divide the probability space of input random variables; (2) Dirac delta function smoothing technique, which replaces the discontinuous Dirac delta function with a continuous Gaussian function. Therefore, the corresponding probability density integral equation formula for the nonlinear vehicle suspension system is:

$$
\begin{aligned}
p_{Y_l}\left(x_l,t\right) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_{\Theta}(\boldsymbol{\theta})\delta\left[x_l - g_l(\boldsymbol{\theta},t)\right]\mathrm{d}\boldsymbol{\theta} \\
&\approx \sum_{q=1}^{N}\left\{\delta\left[x_l - g_l\left(\boldsymbol{\theta}_q,t\right)\right]\int_{\Omega_{\Theta,q}} p_{\Theta}\left(\boldsymbol{\theta}\right)\mathrm{d}\boldsymbol{\theta}\right\} \\
&\approx \sum_{q=1}^{N}\left\{\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\left[x_l-g_l\left(\boldsymbol{\theta}_q,t\right)\right]^2/2\sigma^2}P_q\right\}
\end{aligned}
\tag{17}
$$

Where:

N…The number of representative points with punctuation selected in the probability space;

θ_q…The qth representative point in the probability space;

$\Omega_{\Theta,q}$…The probability space occupied by the qth representative point;

σ…The standard deviation of the Gaussian distribution, and also represents the smoothing parameter;

P_q…The allocation probability of the qth representative point in the probability space.

In the above formula, the first asymptotic expression represents the division of the probability space, which can also be understood as the discretization of the continuous random vector Θ. The second asymptotic expression represents the smoothing of the Dirac function using a Gaussian function with parameter σ. This indicates that the response G obtained through the mapping $x_{l,q} = g_l(\boldsymbol{\theta}_q,t)$ is once again smoothed into a continuous variable.

5 Random response analysis of nonlinear vehicle suspension structure under random road excitation

This paper employs MATLAB to analyze the stochastic response of a seven-degree-of-freedom vehicle system. Parameters of each structural component within the vehicle system are considered to follow a normal distribution [24]. The system's motion differential equations are solved using the ode45 method. When the vehicle travels road surfaces of varying grades at different speeds, the stochastic dynamic responses generated by the random road excitation system serve to validate the applicability of the Direct Probability Integral Method for the stochastic vibration response of nonlinear vehicle suspensions under both random road conditions and structural parameter uncertainties. The complete technical parameters of the vehicle are presented in Table 2. To facilitate clearer observation of the impact of the dispersion in the vehicle's stochastic parameter values on the system response, the coefficient of variation is set to 0.2 for all parameters.

Tab. 2 *Technical parameters of a vehicle*

Parameter	Distribution type	Mean	CV
m_s	Gaussian	1380	0.2
I_y	Gaussian	2440	0.2
I_x	Gaussian	380	0.2
m_1, m_2, m_3, m_4	Gaussian	40.5	0.2
k_1, k_2	Gaussian	17000	0.2
k_3, k_4	Gaussian	22000	0.2
c_1, c_2, c_3, c_4	Gaussian	1500	0.2
$k_{t1}, k_{t2}, k_{t3}, k_{t4}$	Gaussian	192000	0.2
la	Gaussian	1.5	0.2
lb	Gaussian	1.7	0.2
lf	Gaussian	1.8	0.2
lr	Gaussian	1.8	0.2

Furthermore, the classic Monte Carlo Simulation (MCS) method is employed as a comparison to verify the efficiency and accuracy of the Direct Probability Integration Method (DPIM) in solving the response of a nonlinear vehicle suspension system with coupled road excitation and structural parameter randomness. When a sufficient number of sample points are selected for Monte Carlo Simulation, the probability information of the system can be calculated more accurately. The accuracy of MCS predictions increases with the number of points selected. To validate the efficiency and accuracy of DPIM, this paper compares the mean value of the system output response obtained using DPIM with N=1000 representative points against the results from MCS with N=100000 representative points.

When the vehicle travels at a speed of 20m/s on a Grade C road surface, a time-domain model for random road roughness is obtained using the aforementioned methods. As shown in Figure 2.

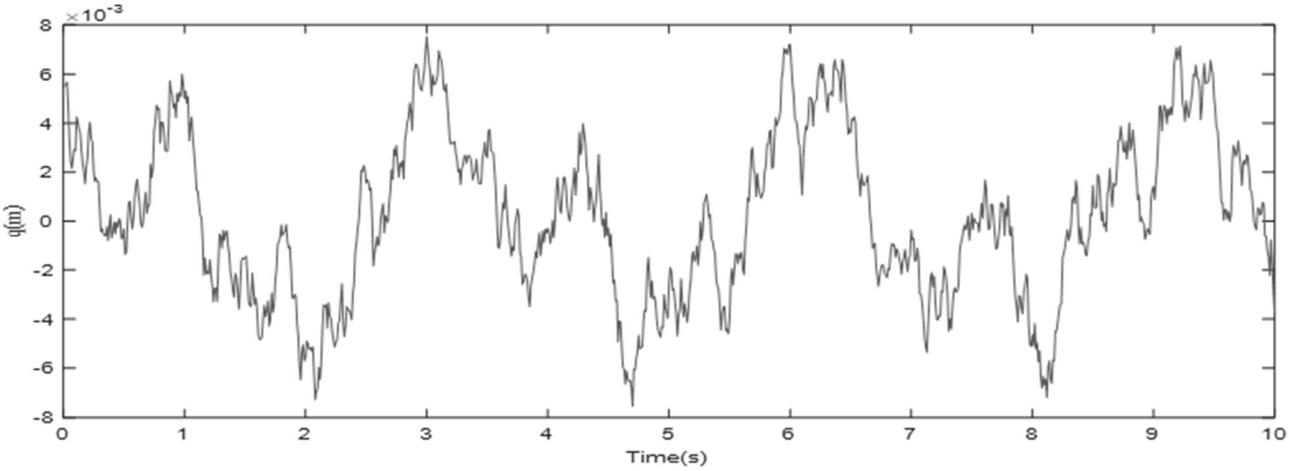

Fig. 2 *Time domain model of random road roughness*

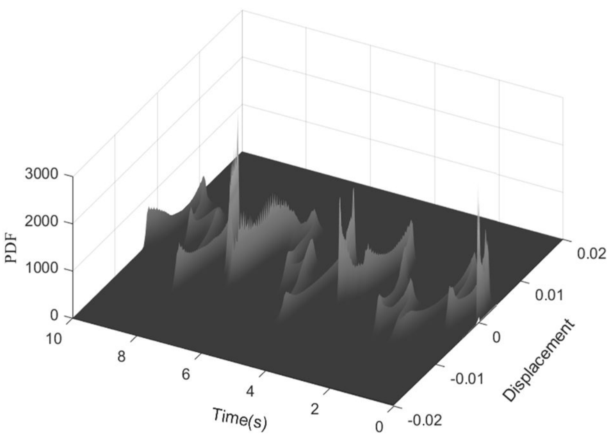

Fig. 3 *Surface of probability density function of vertical displacement of car body*

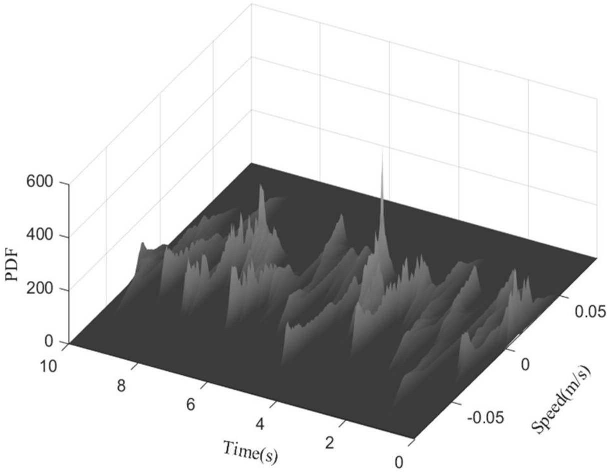

Fig. 4 *Surface of probability density function of vertical speed of vehicle body*

Multiply the above-generated time-domain samples of road roughness with tire stiffness to obtain the random excitation caused by road irregularities. This excitation is used as the random excitation input into the system and substituted into equation (2). The random dynamic response of the nonlinear vehicle suspension system under the coupling of road and structural parameter randomness is solved by the direct probability density integration method. Substituting the obtained random response of vertical displacement and velocity of the vehicle body into equation (7) yields the probability density function surface plots of the vertical displacement and velocity of the vehicle body, as shown in Figures 3 and 4. These probability density function surface plots provide a clearer view of the system's variation under random road excitations.

By observing the probability density function surface of the vehicle's vertical displacement, it can be found that there is little difference in the random response output at the initial moment, t=3s, and t=7s. The peaks of the probability density function surfaces at these three moments are very high, indicating a low vibration frequency of the vehicle body displacement and relatively stable vehicle body. However, at other moments, the differences in response become greater, the peaks of the probability density function images are lower, the vehicle body displacement is more dispersed, and the vehicle vibration is more intense.

By slicing the output probability density surface for the vehicle body displacement at different time points, we can obtain two-dimensional probability density curves for any given moment. Selecting the probability density function curves of vehicle body displacement at t=3s, t=5s, and t=8s, it can be seen that due to different random road excitations, as shown in Figure 5, the PDF images of vehicle body displacement show different variation patterns at different moments.

By analyzing Figure 5, it can be found that even when the representative points selected by DPIM are 100 times fewer than those of MCS, similar results to MCS can still be obtained. The PDFs match well at t=3s, t=8s and t=5s. In complex road excitation conditions, the PDF value of vehicle vertical displacement is smaller, while in smoother road excitation conditions, the PDF value of vehicle vertical displacement is larger. This observation underscores the sensitivity of the vehicle's response to the variability of the road surface, as well as the robustness of the DPIM method in approximating the MCS results with significantly fewer computational resources.

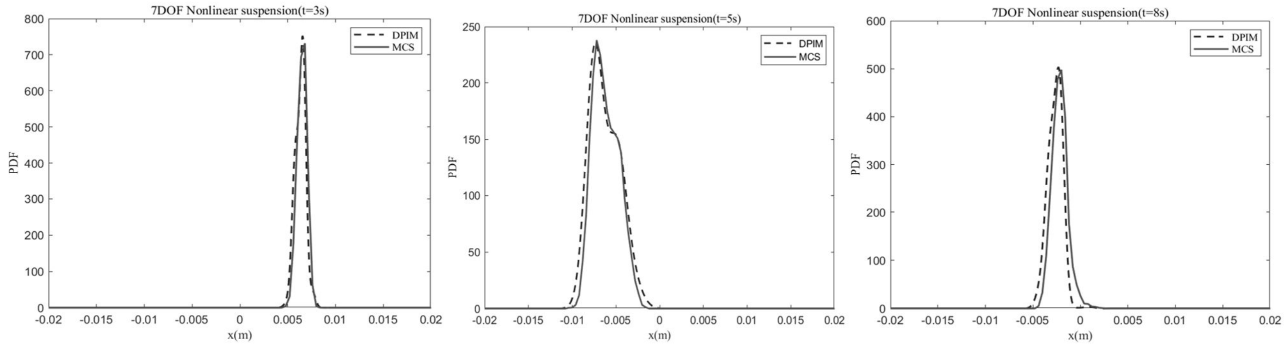

Fig. 5 *Comparison of MCS and DPIM vehicle displacement probability density curves at different times*

Figure 6 compares the mean results of nonlinear vehicle body displacement responses when using DPIM to obtain N=1000 representative samples and MCS to obtain N=100000 samples with punctuation. It can be found that when using the direct probability integral method (DPIM) to calculate the mean of the random response of the system, the accuracy is very high, and the calculation results are basically consistent with the results of multiple simulation sampling by MCS. This illustrates the applicability and efficiency of the DPIM for response analysis of nonlinear vehicle suspension systems under random excitation.

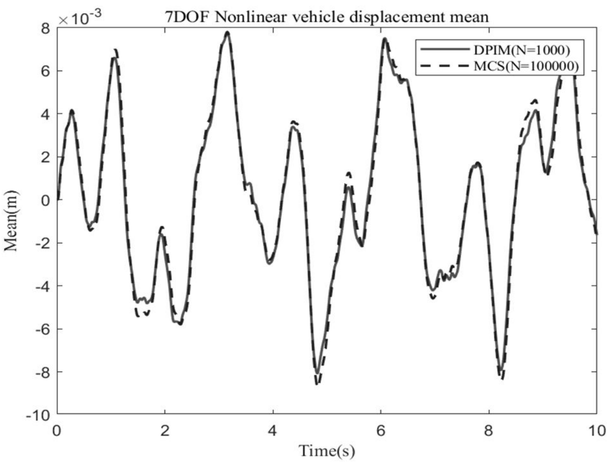

Fig. 6 *Mean displacement of 7DOF nonlinear vehicle body*

When the vehicle travels at a speed of 20 m/s on a Class A road surface, upon encountering the stochastic excitation from the road acting upon the nonlinear suspension system, analysis using the Direct Probability Integral Method (DPIM) yields the probability density surface for the vehicle under this driving condition, as depicted in Figure 7. Comparing Figure 7 with Figure 4, it becomes evident that when the vehicle travels at the same speed but on different grade roads, the values of the probability density function (PDF) for the system's output stochastic response under the random excitation of a Class A road surface are higher than those under the excitation of a Class C road surface. By sectioning Figure 7, three-time probability density function curves are obtained, as shown in Figure 8. When these are contrasted with Figure 5, it is observed that at t=3 seconds, the peak of the probability density function for the vertical displacement of the vehicle body under the excitation of a Class A road is approximately three times that of a Class C road; at t=5 seconds, the peaks differ by about eight times; and at t=8 seconds, the difference is roughly seven times. This observation reveals that under the random excitation of a Class A road surface, the disparity in body displacement is smaller, indicating a more stable vehicle body. This finding also aligns with the fact that a Class A road surface is smoother than a Class C road surface.

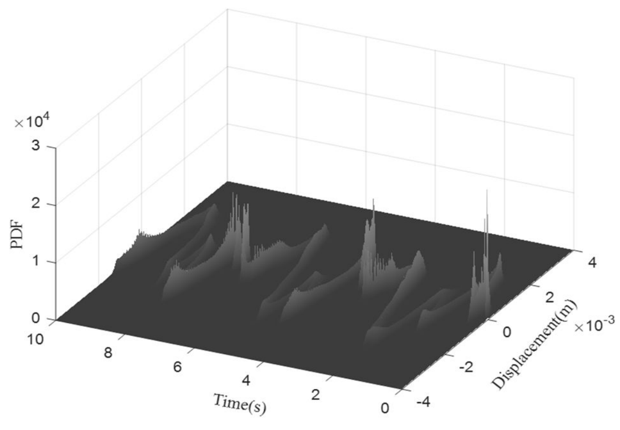

Fig. 7 *Surface of body vertical displacement probability compactness function (Class A road surface)*

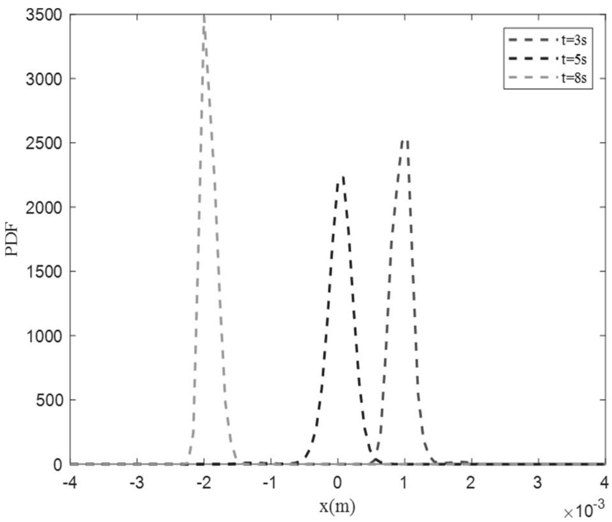

Fig. 8 *Vertical probability density curve of vehicle body at different time*

6 Random response analysis of nonlinear vehicle suspension structure with different parameters

6.1 The influence of different coefficient of variation on the stochastic response of the system

During the manufacturing process of vehicles, structural parameters of the system exhibit randomness due to both technical and human factors. Consequently, the parameters of a nonlinear vehicle suspension system can be viewed as normally distributed, facilitating an investigation into how variations in the coefficients of variation (COV) of these parameters impact the displacement of the vehicle body. By setting the COVs of each structural parameter of the system to 0.1, 0.2, and 0.3 respectively, we employ the Direct Probability Integral Method (DPIM) to analyze the probability density functions (PDFs) of the stochastic responses output by the nonlinear vehicle suspension system under Class C road excitation, considering different levels of structural parameter variability.

Based on Figure 9, it can be observed that changing the variation coefficient of suspension structural parameters leads to significant variations in the peak values of the probability density function (PDF) of vehicle body displacement at different times, and the impact varies depending on the parameter variations of different structures. Generally speaking, the smaller the variation coefficient is, the higher the peak value of the PDF of system vehicle body displacement obtained by the direct probability density integration method, and the smaller the difference in vehicle body displacement. By observing Figures 9a, 9b, and 9c, it can be seen that when the variation coefficient of each structural parameter is 0.1 and 0.2, the PDFs of the random response of vehicle body displacement are similar. When the variation coefficient is 0.3, there is little difference in the PDF of system output vehicle body displacement at the moment when the vehicle is subjected to random road excitation. As the vehicle travels, the peak value of the PDF gradually decreases, and subsequently, there is little difference in the shape and peak value at each moment.

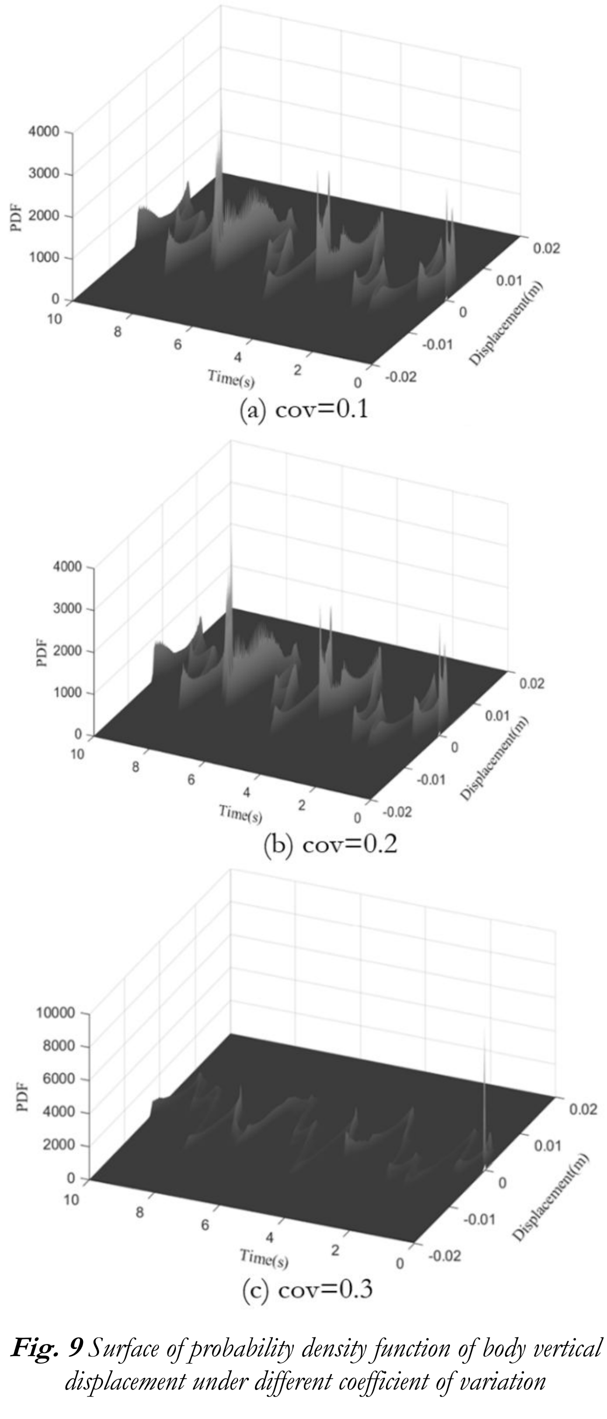

Fig. 9 *Surface of probability density function of body vertical displacement under different coefficient of variation*

By dividing the three-dimensional probability density function (PDF) surface plots of vehicle body displacement obtained under different coefficients of variation along the time axis, probability density function curves at different times can be obtained. By selecting the time points of t=1s and t=5s for dissection, the PDFs of vehicle body displacement with different coefficients of variation at 1s and 5s can be obtained, as shown in Figure 10. It clearly shows that as the coefficient of variation increases, the peak value of the PDF of the system output vehicle body displacement gradually decreases under the same random road excitation, and the range of the entire vehicle body displacement gradually increases. This indicates that the larger the errors in the structural parameters of the nonlinear vehicle suspension system, the lower the stability of the entire system, and stronger vibrations may occur when subjected to external excitations. Therefore, the uncertainty of vehicle structural parameters is an important factor that cannot be ignored in terms of its impact on vehicle vibrations.

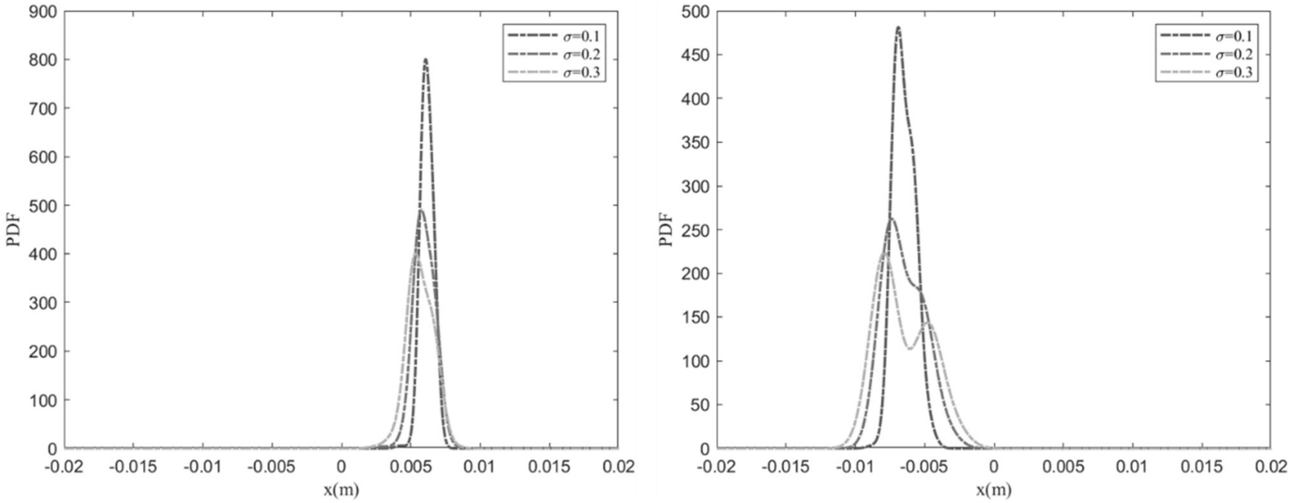

Fig. 10 *Probability density curves with different coefficient of variation at 1s and 5s*

6.2 The influence of nonlinear strength on stochastic response of system

During vehicular operation, the nonlinearity of spring stiffness and the coupling effects between various components are indispensable considerations. Changes in the intensity of spring stiffness nonlinearity simultaneously impact the stability of the vehicle body. Therefore, analyzing the effect of varying degrees of suspension spring nonlinearity on body displacement under random road excitation is of paramount importance. When a vehicle travels at a speed of 20 m/s on a Class C road surface, modifying the intensity of nonlinearity in the suspension spring stiffness, the Direct Probability Integral Method (DPIM) can be employed to obtain probability density functions (PDFs) for body displacement under different levels of nonlinearity [25]. This enables us to assess the implications for vehicular stability during travel.

As previously mentioned, the intensity of nonlinearity in the vehicle's spring suspension stiffness can be characterized by a cubic displacement relationship, with variations in the degree of nonlinearity achieved through adjustments to the parameter. By setting this parameter to values of 0.1, 10, 10e2, 10e3, and 10e4, the Direct Probability Integral Method (DPIM) is used to derive mean curves for body displacement under differing intensities of nonlinear spring stiffness, as illustrated in Figure 11. From the figure, it is evident that when the value of is 0.1 and 10, the mean body displacement of the vehicle's nonlinear suspension system under random road roughness excitation aligns well, showing essential consistency. As the intensity of nonlinear spring stiffness increases to 10e2 and 10e3, there is a noticeable discrepancy in the mean body displacement at inflection points, with the absolute value of displacement at these points becoming larger as the degree of nonlinearity grows. When the nonlinearity degree parameter of the vehicle suspension system's spring stiffness is set to 10e4, the overall mean body displacement exhibits significant deviations. In regions where the change in mean body displacement is small, the amplitude of change in mean body displacement becomes notably larger. Overall, the stronger the nonlinearity in spring stiffness, the more intense the body displacement vibration becomes.

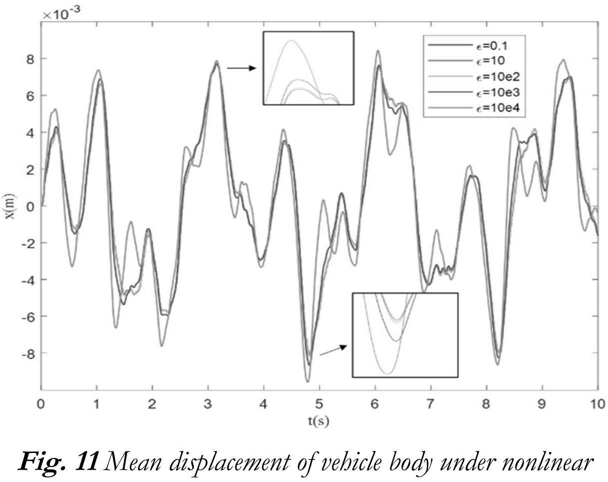

Fig. 11 *Mean displacement of vehicle body under nonlinear strength with different suspension spring stiffness*

Through the direct probability density integral method, we can obtain the probability density surface of the displacement of the vehicle under different nonlinear strength of the suspension spring under the excitation of random road roughness, and then analyze the vibration law of the vehicle by the probability density curve at each moment. The probability density function surfaces of different nonlinear intensities are shown in Figure 12.

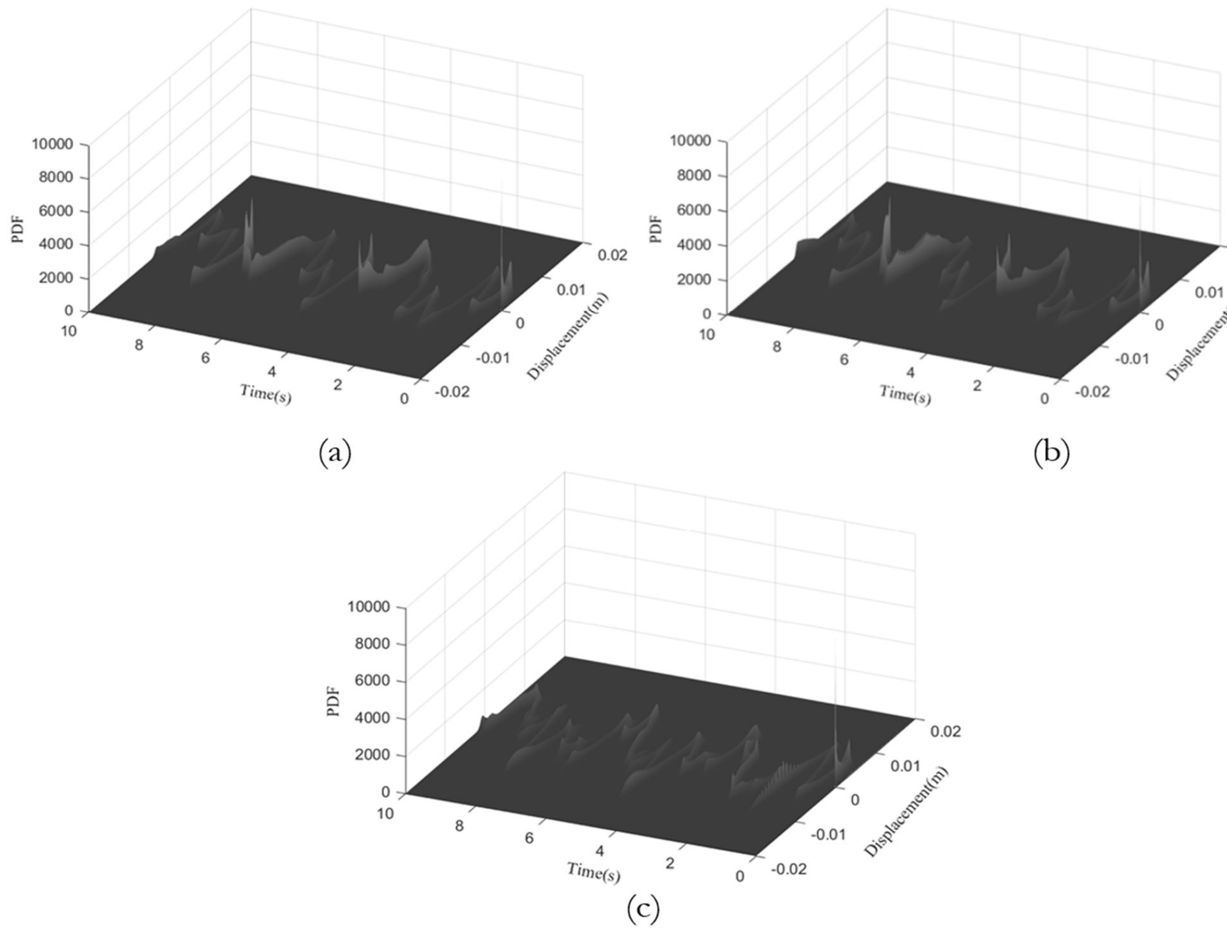

Fig. 12 *Probability density function of vehicle displacement under different nonlinear strengths*

Through the probability density surface plot, it can be seen that when the nonlinearity parameter of spring stiffness is set to 10e2 and 10e3, as shown in Figures 12a and b, the probability density function surfaces of the vehicle body displacement are consistent for both values. At the initial moment and at 7s, the peak of the probability density function is higher, indicating smaller differences in vehicle body displacement and smoother driving. When the nonlinearity parameter of the vehicle suspension spring stiffness is 10e4, Figure 12c shows that the peak of the probability density function is higher at the initial moment. However, as time increases, the peak of the probability density function decreases under random road excitations, showing significant differences from Figures 12a and b. The difference in peak heights becomes smaller, and the overall trend is relatively stable. By dividing the probability density function surface plot, the probability density function curve at each moment can be obtained, facilitating a better analysis of the unique vibration state of the vehicle body under different nonlinear strengths of the suspension spring. Figure 13 shows the probability density function curves of vehicle body displacement under different nonlinear strengths at 1s and 5s, respectively. As seen from Figure 13, under random road excitations, when the suspension spring stiffness exhibits soft nonlinearity, the probability density function of vehicle body displacement changes less. As the nonlinear strength increases, the probability density function curve of vehicle body displacement shifts to the right, indicating an increase in overall vehicle body displacement. When the nonlinear strength of the vehicle continues to increase, the vehicle body displacement changes significantly, resulting in more intense vehicle vibration. These findings emphasize the intricate relationship between spring stiffness nonlinearity and vehicular dynamics, highlighting the need for precise parameterization to achieve optimal performance and ride quality, especially under stochastic environmental conditions.

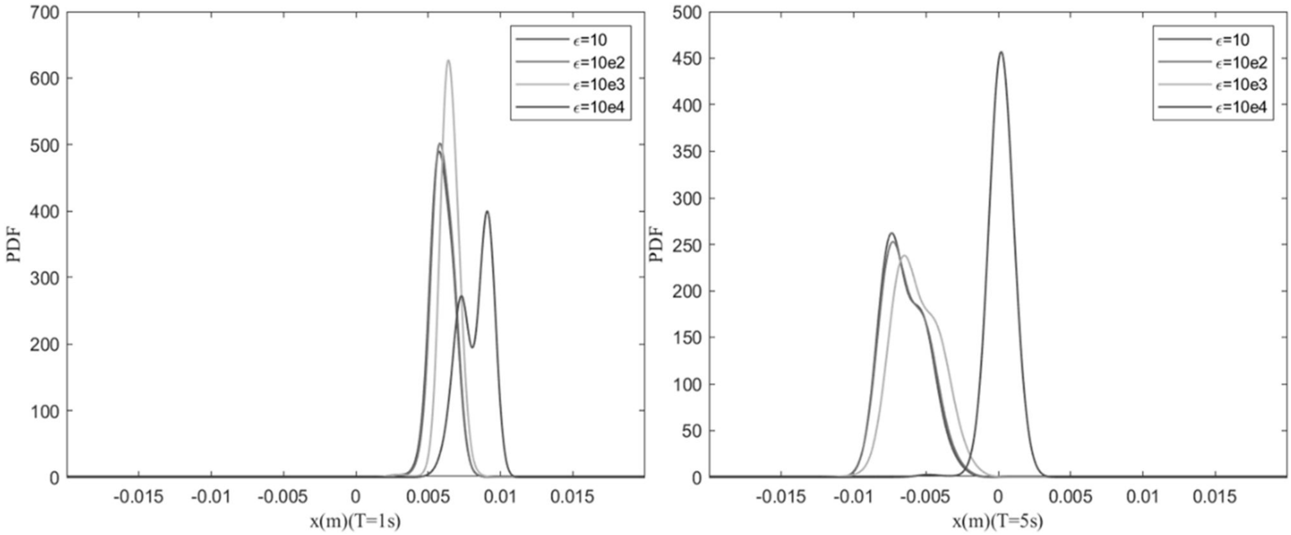

Fig. 13 *Curves of vehicle displacement probability density function with different nonlinear strengths at 1s and 5s*

7 Conclusion

In this paper, a seven-degree-of-freedom nonlinear suspension model for the whole vehicle is established, and the dynamic equation of the suspension system under the coupling of road surface and structural parameter randomness is constructed. Based on the conservation of probability, the direct probability density integration method is used to analyze the random dynamic response of the output of the nonlinear vehicle suspension system with random road surface and random structural parameters. In addition, the variation coefficient of structural parameters and the nonlinear strength of suspension spring stiffness are taken as key parameters affecting vehicle dynamic behavior. The effects of different variation coefficients of structural parameters and different nonlinear strengths of suspension springs on the random dynamic response of vehicle body displacement are studied respectively. The main conclusions of this paper are as follows:

- (1) By comparing the direct probability density integration method with the currently more mature Monte Carlo simulation method, it is verified that the direct probability density integration method can be applied to nonlinear vehicle suspension systems with coupled randomness of road surface and structural parameters. The results are highly accurate, the calculation time is reduced, the computational efficiency is improved, reflecting the advantages of the direct probability density integration method.
- (2) When a vehicle travels at the same speed on different grades of roads, the probability density functions (PDFs) of the random responses output by the nonlinear vehicle suspension system due to excitations from random road surfaces are different. Comparing the PDFs of the system's random responses obtained through the direct probability density integration method when the vehicle travels on Grade A and Grade C roads, respectively, it is found that the peak of the PDF at each moment under Grade A road excitation is much higher than that under Grade C road excitation, and the probability density function curve at each moment is shifted to the left. The smaller the difference in vehicle body displacement at each moment under Grade A road excitation, the smoother the vehicle travels.
- (3) Based on this, the analysis of the random dynamic response of nonlinear vehicle body displacement was conducted by studying different variation coefficients of structural parameters. The results show that when the variation coefficient of the structural parameters of the suspension system is 0.1, the peak value of the PDF of the vertical displacement of the vehicle body at various moments is approximately two orders of magnitude higher compared to the PDFs with variation coefficients of 0.2 and 0.3, indicating smoother vehicle travel. However, as the variation coefficient increases, the peak value of the probability density function curve decreases, indicating

greater variability in vehicle body displacement at a given moment and more intense vibration.

- (4) Under the excitation of random road roughness, the vibration state of the vehicle body was observed by varying the stiffness of the springs in the nonlinear suspension. In the case of soft nonlinearity, the probability density function curves of vehicle body displacement at various moments are relatively consistent. As the nonlinear strength increases, the vehicle body displacement becomes larger. When stronger nonlinear spring stiffness is adopted, the probability density function continues to shift to the right, indicating a continued increase in vehicle body displacement. This leads to a decrease in the vibration isolation performance of the suspension system, increasing the bumpy ride and affecting passenger comfort.

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