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Analytic Solution of the Motion Equations of the Rolling Stock Chassis Incorporating the Effect of Asymmetry and Kinematics Excitation

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The article provides an analytical solution for the dynamics of a vehicle chassis designed for both road and rail operation, featuring either single or multiple primary linear suspensions using coil springs. It derives the equations of motion for a simplified two-axle chassis model that includes both a basic primary suspension and a simplified chassis suspension. The study focuses on the simplest calculation model to analyze suspension behavior, taking into account the asymmetry in spring stiffness and geometric positioning. There is an unequal distribution of weight across the vehicle body. An analysis is conducted on a comprehensive vehicle model with nine degrees of freedom. The analytical solution is obtained using Lagrange equations of the second kind, alongside various calculation techniques such as Laplace transformation. Due to the scope of the article, calculations of all coefficients of the matrices are not presented, but a link to other works of the authors is given, where these procedures are presented. The proposed analytical solution makes it possible to derive an effective algorithm for the application of computer technology. The use of the proposed procedures allows determining the permissible asymmetry of vehicles for safe driving, taking into account structural asymmetry, kinematic excitation asymmetry (always occurs) and suspension asymmetry (almost always occurs).

Keywords: Vibration, Motion Equations, Rolling stock

1 Introduction

Accidents occur in road and rail traffic, the causes of which are not easily explained. The cause may be various influences of both a structural and operational nature, e.g. asymmetry of weight distribution, asymmetry and properties of suspension and dissipative elements of vehicles, asymmetry of kinematic excitation of vertical vibrations when crossing unevenness, etc. [1, 6].

These facts, which are always present in vehicles, were the subject of an investigation into the impact of asymmetry affecting the vertical oscillations of wheeled vehicles (road and rail) at different kinematic excitation. Both structural and operational asymmetry (asymmetric weight distribution in the construction and loading of the vehicles) was examined. This asymmetry, combined with the uneven kinematic excitation of vertical oscillations (due to crossing irregular surfaces), can, in extreme cases, cause the wheel to lose contact with the road or rail surface, potentially leading to an accident. The structural design usually assumes complete symmetry of the vehicle. But this assumption cannot be maintained during design or in operation (asymmetrical distribution of the vehicle and load weight related to the centre of gravity, asymmetry of the kinematic excitation, etc.). This causes a number of operational problems.

It can be found an enormous literature on this issue (e.g., [2, 3]), but the effect of asymmetry in spatial vehicle models is absent in these works. However, only partial solutions are provided, such as addressing the effects of asymmetry in spatial models using half-plane models (which assume planar symmetry along the vehicle's longitudinal axis). These models may account for factors like the displacement of the center of gravity from the center of the axle spacing, variations in suspension stiffness, viscous damping, or combinations of these factors, but always maintain symmetry along the longitudinal axis [2-5]. Most often, the problem of vehicle vibration is solved on quarter models (two axes of symmetry, longitudinal and transverse, are assumed), with different numbers of bodies, which are usually vertically elastically coupled, with dissipative elements. The models feature additional degrees of freedom with coupled displacements in vertical direction. However, these models cannot replace the effect of general asymmetry, which has not yet been satisfactorily resolved.

Given these considerations, it is crucial to examine how asymmetry influences the vertical oscillation of vehicles by exploring the different sources and consequences of excitation through spatial modeling. These models should incorporate key types of asymmetry relative to the vehicle's geometric reference axes—namely, the perpendicular axes of the wheelbase and track. In particular, the relevant asymmetries include:

- The distribution of the vehicle's mass with respect to its geometric axes of symmetry, encompassing the location of the center of gravity and the alignment of the principal central axes of inertia for both the unloaded structure and the fully loaded vehicle.
- The geometry of the arrangement of elastic and dissipative elements in the connections between individual bodies of the vehicle system, along with their mechanical properties (such as spring stiffness and viscous damping intensity), assumes small displacements and rotations of the system's components, and between individual variables the linear couplings.
- Kinematic excitation, referring to the irregularities in the road surface or track profile that generate system excitation at the contact interface (wheel–vehicle or wheel–track).These types of asymmetries may occur individually or in combination; in practice, however, the third case is almost always present.

2 Methodology

Currently, FEM methods (Simpac, Adams, Alaska, etc.) are most often used to solve vehicle dynamics, which, although they give satisfactory results, do not allow the analysis of the influence of individual types of asymmetry. The analytical solution, on the other hand, allows a better understanding of the processes that occur during vertical oscillation due to asymmetry, either individually or cumulatively.

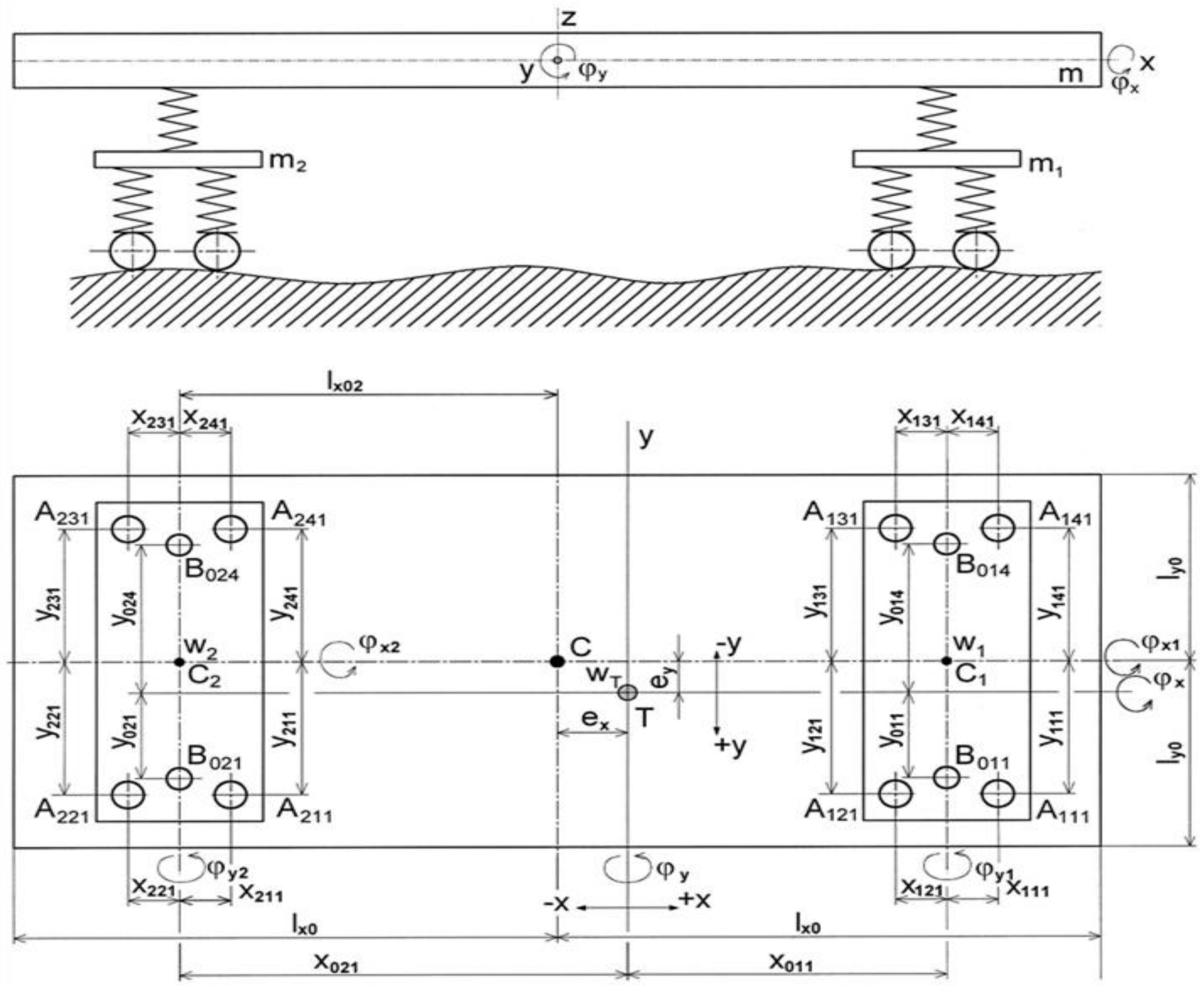

Fig. 1 *Simple spatial model of the wheeled vehicle chassis (four-axle)*

Where:

C…Centre of geometry;

T…Centre of gravity;

A_{jkl}…Springs of primary suspension (j = 1, 2 - mass m_1, m_2, k = 1, 2, 3, 4 - quadrant, i = 1, ... n – order number of springs in case of multisuspension, our case i = 1);

B_{jki}…Springs of secondary suspension (j = 0 – body mass m, k = 1, 2 - order number of chassis 1, or 2, i = 1, 4 - mid-point affiliation A_{111}, A_{141}, or A_{211}, A_{241},

e_x, e_y…Deviation of gravity centre;

w_T, w_1, w_2…Vertical displacement of gravity centre;

φ_x, φ_y, φ_{x1}, φ_{y1}, φ_{x2}, φ_{y2}…Rotation of the body around the central axes of inertia of the masses m, m_1, m_2.

The analytical solution of the effect of asymmetry in the wheeled vehicle chassis required the development of an appropriate and simplified spatial model (Fig. 1). This model was employed to analyze various cases of asymmetry, enabling the comparison and validation of results obtained through different approaches—experimental as well as theoretical (including analytical, numerical, and simulation methods). The vehicle body is represented as a stiff flat bodywith mass m, considering an non-uniform distribution of mass (the center of gravity is offset from the geometric center by e_x and e_y). This asymmetry causes a rotation of the principal central axes of inertia relative to the axes of geometric symmetry. The chassis are represented by rigid plates with masses m_1 and m_2, assuming symmetric mass distribution, i.e., the center of gravity $T_1 \equiv C_1$ and $T_2 \equiv C_2$. The principal central axes of inertia are aligned with the geometric symmetry axes. As shown in Fig. 1, this system possesses nine degrees of freedom.

3 Solution and Discussion

The solution assumes body stiffness, small displacements and rotations, and linear spring characteristics. It considers only the vertical displacement of any arbitrary point on a body (m, m_1 m_2) determined by the change in the position of the body's center of gravity (w, w_1,w_2)) and the displacement caused by the rotations φ_x and φ_y, φ_{x1} and φ_{y1}, φ_{x2} and φ_{y2} of the principal axes of inertia of the masses (m, m_1 m_2), together with the distances of these masses from the reference point. The geometry of the secondary spring and the asymmetry in the stiffness parameters are also taken into account.

Determination of oscillation in vertical direction were derived using Lagrange's equations.

Kinetic energy of bodies:

$$\begin{aligned} E_k = \frac{1}{2}\cdot m\cdot \dot{w}^2 + \frac{1}{2}\cdot\left(J_x\cdot\dot{\varphi}_x^2 + J_y\cdot\dot{\varphi}_y^2 - 2\cdot D_{xy}\cdot\dot{\varphi}_x\cdot\dot{\varphi}_y\right) + \frac{1}{2}\cdot m_1\cdot\dot{w}_1^2 + \frac{1}{2}\cdot J_{x1}\cdot\dot{\varphi}_{x1}^2 + \frac{1}{2}\cdot J_{y1}\cdot\dot{\varphi}_{y1}^2 + \\ + \frac{1}{2}\cdot m_2\cdot\dot{w}_2^2 + \frac{1}{2}\cdot J_{x2}\cdot\dot{\varphi}_{x2}^2 + \frac{1}{2}\cdot J_{y2}\cdot\dot{\varphi}_{y2}^2 \end{aligned} \tag{1}$$

Where:

J_x, J_y, J_{x1}, J_{y1}, J_{x2}, J_{y2}…Moments of inertia of the body and chassis to the central axes of the individual bodies;

m, m_1, m_2…Mass of bodies (frame, chassis);

D_{xy}…Frame deviation moment;

w, $\dot{w}_1$, $\dot{w}_2$…Velocities of displacement of the centres of gravity of bodies;

$\dot{\varphi}_x$, $\dot{\varphi}_y$, $\dot{\varphi}_{x1}$, $\dot{\varphi}_{y1}$, $\dot{\varphi}_{x2}$, $\dot{\varphi}_{y2}$…Angular velocity of rotation of masses.

Potential energy E_p of chassis models m_1 and m_2 and frame models m.

$$E_p = \frac{1}{2}\cdot\sum_{j=1}^{2}\sum_{k=1}^{4}\sum_{i=1}^{k_j} k_{jki}\cdot w_{jki}^2 + \frac{1}{2}\cdot\sum_{j=0}\sum_{k=1,2}\sum_{i=1,4} k_{jki}\cdot w_{jki}^2 \tag{2}$$

Following relations are representing the vertical displacements of chassis springs supports.

Body m_1, j = 1

Point	Vertical displacement	Stiffness constant	
A_{111}	$w_{111} = w_1 - y_{111}\cdot\varphi_{x1} + x_{111}\cdot\varphi_{y1} - h_{111}$	k_{111}	
A_{121}	$w_{121} = w_1 - y_{121}\cdot\varphi_{x1} - x_{121}\cdot\varphi_{y1} - h_{121}$	k_{121}	(3a)
A_{131}	$w_{131} = w_1 + y_{131}\cdot\varphi_{x1} - x_{131}\cdot\varphi_{y1} - h_{131}$	k_{131}	
A_{141}	$w_{141} = w_1 + y_{141}\cdot\varphi_{x1} + x_{141}\cdot\varphi_{y1} - h_{141}$	k_{141}	

Body m_2, j = 2

Point	Vertical displacement	Stiffness constant	
A_{211}	$w_{211} = w_2 - y_{211}\cdot\varphi_{x2} + x_{211}\cdot\varphi_{y2} - h_{211}$	k_{211}	
A_{221}	$w_{221} = w_2 - y_{221}\cdot\varphi_{x2} - x_{221}\cdot\varphi_{y2} - h_{221}$	k_{221}	(3b)
A_{231}	$w_{231} = w_2 + y_{231}\cdot\varphi_{x2} - x_{231}\cdot\varphi_{y2} - h_{231}$	k_{231}	
A_{241}	$w_{241} = w_2 + y_{241}\cdot\varphi_{x2} + x_{241}\cdot\varphi_{y2} - h_{241}$	k_{241}	

In the case of the above model (Fig. 1), the body m is suspended in the chassis axis by one spring on each side.

To designate the individual points where the springs act. The numbering system employed to divide the chassis into four sections is not applicable in this case. Instead, the impact points in springs positions on the chassis are labeled as B_{jki}, with the coordinates of these points represented as x_{jki}, y_{jki}. Here, $j = 0$ refers to the chassis, $k = 1$ or 2 refers to the numbering of the chassis (1 or 2), and $i = 1$ or 4 corresponds to the geometry half and points A_{111} and A_{141}, respectively.

Vertical displacement of frame:

Point	Vertical displacement	Stiffness constant	
B_{011}	$w_{011}= w- y_{011}\cdot\varphi_x+x_{011}\cdot\varphi_y- w_1+(y_{011} + e_y)\cdot\varphi_{x1}$	k_{011}	
B_{014}	$w_{014}= w+ y_{014}\cdot\varphi_x+x_{014}\cdot\varphi_y- w_1+(y_{014} - e_y)\cdot\varphi_{x1}$	k_{014}	(4)
B_{021}	$w_{021}= w- y_{021}\cdot\varphi_x-x_{021}\cdot\varphi_y- w_2+(y_{021} + e_y)\cdot\varphi_{x2}$	k_{021}	
B_{024}	$w_{024}= w+ y_{024}\cdot\varphi_x-x_{024}\cdot\varphi_y- w_2-(y_{024} - e_y)\cdot\varphi_{x2}$	k_{024}	

After modification, the relations (3a, 3b, 4) are substituted into (1) and (2), which are also modified and substituted into Lagrange's equations of the second order

$$\frac{d}{dt}\cdot\frac{\partial E_k}{\partial \dot{q}_j}-\frac{\partial E_k}{\partial q_j}+\frac{\partial E_p}{\partial q_j}=Q_j \tag{5}$$

Where:

q_j…Generalized coordinates for $j = 1, 2 …. 9$, ($q_j = w, w_1, w_2, \varphi_x, \varphi_y, \varphi_{x1}, \varphi_{y1}, \varphi_{x2}, \varphi_{y2}$);

Q_j…Generalized forces, including kinematics excitation.

After substitution, arranging and performing the appropriate derivations according to the individual coordinates, the equations of motion are obtained in matrix form.

$$\begin{bmatrix} J_x & -D_{xy} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -D_{xy} & J_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{x1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{y1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{x2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{y2} \end{bmatrix} \cdot \begin{bmatrix} \ddot{\varphi}_x \\ \ddot{\varphi}_y \\ \ddot{w} \\ \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{\varphi}_{x1} \\ \ddot{\varphi}_{y1} \\ \ddot{\varphi}_{x2} \\ \ddot{\varphi}_{y2} \end{bmatrix} + \begin{bmatrix} \varkappa_{11} & \varkappa_{12} & \varkappa_{13} & \varkappa_{14} & \varkappa_{15} & \varkappa_{16} & 0 & \varkappa_{18} & 0 \\ \varkappa_{21} & \varkappa_{22} & \varkappa_{23} & \varkappa_{24} & \varkappa_{25} & \varkappa_{26} & 0 & \varkappa_{28} & 0 \\ \varkappa_{31} & \varkappa_{32} & \varkappa_{33} & \varkappa_{34} & \varkappa_{35} & \varkappa_{36} & 0 & \varkappa_{38} & 0 \\ \varkappa_{41} & \varkappa_{42} & \varkappa_{43} & \varkappa_{44} & 0 & \varkappa_{46} & \varkappa_{47} & 0 & 0 \\ \varkappa_{51} & \varkappa_{52} & \varkappa_{53} & 0 & \varkappa_{55} & 0 & 0 & \varkappa_{58} & \varkappa_{59} \\ \varkappa_{61} & \varkappa_{62} & \varkappa_{63} & \varkappa_{64} & 0 & \varkappa_{66} & \varkappa_{67} & 0 & 0 \\ 0 & 0 & 0 & \varkappa_{74} & 0 & \varkappa_{76} & \varkappa_{77} & 0 & 0 \\ \varkappa_{81} & \varkappa_{82} & \varkappa_{83} & 0 & \varkappa_{85} & 0 & 0 & \varkappa_{88} & \varkappa_{89} \\ 0 & 0 & 0 & 0 & \varkappa_{95} & 0 & 0 & \varkappa_{98} & \varkappa_{99} \end{bmatrix} \cdot \begin{bmatrix} \varphi_x \\ \varphi_y \\ w \\ w_1 \\ w_2 \\ \varphi_{x1} \\ \varphi_{y1} \\ \varphi_{x2} \\ \varphi_{y2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{bmatrix} \tag{6}$$

Where:

D_{xy}…Deviation moment;

J_x, J_y…Moments of inertia related to appropriate axes;

$\varkappa_{ij}$…Stiffness matrix elements (they are determined by the stiffness constants k_{jki}, dimensions of support geometry x_{jki}, y_{jki} and eccentricity e_x and e_y);

Q_i…Kinematics excitation function (they are determined by the stiffness constants of chassis springs k_{jki} and functions of displacement in time $h(t)$ in the position of wheel spring support).

Elements of stiffness matrix $\varkappa_{ij}$ are (see above) functions of constants k_{jki}, dimensions of support geometry x_{jki}, y_{jki} and eccentricity e_x and e_y – see Fig. 1. Calculation of particular elements $\varkappa_{ij}$ is beyond the scope of the article and it is presented e.g. [1].

Right side functions (6) of kinematics excitation are determined by relations:

$$Q_1 = 0 \qquad Q_2 = 0 \qquad Q_3 = 0$$

$$\begin{aligned} Q_4 &= k_{111}\cdot h_{111}+k_{121}\cdot h_{121}+k_{131}\cdot h_{131}+k_{141}\cdot h_{141} \\ Q_5 &= k_{211}\cdot h_{211}+k_{221}\cdot h_{221}+k_{231}\cdot h_{231}+k_{241}\cdot h_{241} \\ Q_6 &= -k_{111}\cdot h_{111}\cdot y_{111}-k_{121}\cdot h_{121}\cdot y_{121}+k_{131}\cdot h_{131}\cdot y_{131}+k_{141}\cdot h_{141}\cdot y_{141} \\ Q_7 &= k_{111}\cdot h_{111}\cdot x_{111}-k_{121}\cdot h_{121}\cdot x_{121}-k_{131}\cdot h_{131}\cdot x_{131}+k_{141}\cdot h_{141}\cdot x_{141} \\ Q_8 &= -k_{211}\cdot h_{211}\cdot y_{211}+k_{221}\cdot h_{221}\cdot y_{221}-k_{231}\cdot h_{231}\cdot y_{231}+k_{241}\cdot h_{241}\cdot y_{241} \\ Q_9 &= k_{211}\cdot h_{211}\cdot x_{211}-k_{221}\cdot h_{221}\cdot x_{221}-k_{231}\cdot h_{231}\cdot \mathrm{y}_{231}+k_{241}\cdot h_{241}\cdot x_{241} \end{aligned} \tag{7}$$

Where:

$h_{jki} = k_{jki}(t)$…The temporal profile of the kinematic excitation at each axle position and their sequence.

For further solution or general formulation of its procedure and creation of an algorithm suitable for computer processing of the generalized system (6) it is necessary to modify it into the form

$$M_h \cdot \ddot{q}_j + M_k \cdot q_j = Q_j(t) \qquad (8)$$

Where:

$M_h(\alpha_{ij})$…Mass matrix;

M_k…Stiffness matrix;

Q_j…Kinematic excitation matrix.

Multiplying equation (8) by the diagonal matrix $D(d_{ij})$ from left side, where $d_{ij} = 1/a_{ij}$ – diagonal elements of mass matrix $M_h(\alpha_{ij})$, then

$$d_{11} = J_x^{-1} \quad d_{22} = J_y^{-1} \quad d_{33} = m^{-1} \quad d_{44} = m_1^{-1} \quad d_{55} = m_2^{-1}$$
$$d_{66} = J_{x1}^{-1} \quad d_{77} = J_{y1}^{-1} \quad d_{88} = J_{x2}^{-1} \quad d_{99} = J_{y2}^{-1}$$

We get the equation in form:

$$M \cdot \ddot{q}_j + K \cdot q_j = F_j(t) \qquad (9)$$

Where:

M…Mass matrix (= DM_h);

K…Stiffness matrix (= DM_k);

$Fj(t)$…Forces vector (= $d_{ij}Q_j$, where $i = j$).

Mass matrix is possible to express in form M = E+S and equation (9) is transformed into the form:

$$(E+S) \cdot \ddot{q}_j + K \cdot q_j = Q_j(t) \qquad (10)$$

Where:

E…Unit matrix;

S…Mass matrix.

Elements s_{ij} of mass matrix S are determined by equations $s_{11} = s_{22} = 0$, $s_{ij}=0$ for $i = 3, \dots 9; j = 3, \dots .9$, and $s_{12} = -D_{xy}/J_x$, $s_{21} = -D_{xy}/J_y$ determine the impact of a mass distribution which is asymmetrical. Specifically, this effect can involve the angular displacement of the principal axes of inertia with respect to the central axes that are parallel to the geometric symmetry axes of the body, such as those of a plate or a vehicle.

Similarly, the elements a_{ij} of the matrix K are determined by dividing the i-th row of the stiffness matrix by the i-th element on the diagonal of the inertia matrix [1] (the elements a_{ij} are chosen $a_{ij} \neq 0$ for generality reasons). Similarly, the excitation functions $F_j(t)$ are determined from the functions $Qj(t)$ and the vector of excitation forces is also chosen $F_j(t) \neq 0$for generality reasons.

The required quantities of displacements and rotations are denoted by generalized coordinates in the above relations as follows: $\varphi_x(t) \rightarrow q_1(t)$, $\varphi_y(t) \rightarrow q_2(t)$, $w(t) \rightarrow q_3(t)$, $w_1(t) \rightarrow q_4(t)$, $w_2(t) \rightarrow q_4(t)$, $\varphi_{x1}(t) \rightarrow q_5(t)$, $\varphi_{y1}(t) \rightarrow q_6(t)$, $\varphi_{x2}(t) \rightarrow q_7(t)$, $\varphi_{y2}(t) \rightarrow q_9(t)$.

In a symmetrical arrangement of the system

$$k_{111} = k_{121} = k_{131} = k_{141} = k_{211} = k_{221} = k_{231} = k_{241} = k_{12}$$
$$x_{111} = x_{121} = x_{131} = x_{141} = x_{211} = x_{221} = x_{231} = x_{241} = x_{12}$$
$$y_{111} = y_{121} = y_{131} = y_{141} = y_{211} = y_{221} = y_{231} = y_{241} = y_{12}$$
$$k_{011} = k_{014} = k_{021} = k_{024} = k_0$$
$$x_{011} = x_{014} = x_{021} = x_{024} = x_0$$
$$y_{011} = y_{014} = y_{021} = y_{024} = y_0$$
$$e_x = 0 \qquad e_y = 0 \qquad D_{xy} = 0 \rightarrow s_{12} = s_{21} = 0$$

and system of motion equations (10) is transformed into the form:

$$E \cdot \ddot{q}_j + K \cdot q_j = F_j \qquad \text{for } j = 1, 2, \dots 9 \qquad (11)$$

Where stiffness matrix

$$K = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} & 0 & a_{18} & 0 \\ 0 & a_{22} & 0 & a_{24} & a_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & a_{35} & 0 & 0 & 0 & 0 \\ 0 & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{52} & a_{53} & 0 & a_{55} & 0 & 0 & 0 & 0 \\ a_{61} & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} & 0 & 0 \\ a_{81} & 0 & 0 & 0 & 0 & 0 & 0 & a_{88} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{99} \end{bmatrix}$$

The elements of the stiffness matrix depend on the spring stiffness values k_{12}, k_0 and the positions x_{12}, y_{12}, x_0, y_0. Assuming a fully symmetric system of elastically supported and constrained bodies (Fig. 1), The original system of nine coupled differential equations simplifies to seven coupled differential equations along with two separate harmonic motion equations for q_7 and q_9.

$$\ddot{q}_7 + a_{77}q_7 = F_7(t) \qquad \ddot{q}_9 + a_{99}q_9 = F_9(t) \qquad (12)$$

Numerical methods, analytical solutions using matrix calculus, Lagrange's method of variation of constants, or the Laplace transform can be employed to solve the system of inhomogeneous differential equations. (8).

When we apply the Laplace transform (where initial conditions are equal to zero), we obtain a system of linear algebraic equations [1].

$$\begin{bmatrix} p^2+a_{11} & s_{12}p+a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ s_{21}p+a_{21} & p^2+a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ a_{31} & a_{32} & p^2+a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ a_{41} & a_{42} & a_{43} & p^2+a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\ a_{51} & a_{52} & a_{53} & a_{54} & p^2+a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & p^2+a_{66} & a_{67} & a_{68} & a_{69} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & p^2+a_{77} & a_{78} & a_{79} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & p^2+a_{88} & a_{89} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & p^2+a_{99} \end{bmatrix} \cdot \begin{bmatrix} \bar{y}_1(p) \\ \bar{y}_2(p) \\ \bar{y}_3(p) \\ \bar{y}_4(p) \\ \bar{y}_5(p) \\ \bar{y}_6(p) \\ \bar{y}_7(p) \\ \bar{y}_8(p) \\ \bar{y}_9(p) \end{bmatrix} = \begin{bmatrix} \bar{F}_1(p) \\ \bar{F}_2(p) \\ \bar{F}_3(p) \\ \bar{F}_4(p) \\ \bar{F}_5(p) \\ \bar{F}_6(p) \\ \bar{F}_7(p) \\ \bar{F}_8(p) \\ \bar{F}_9(p) \end{bmatrix} \qquad (13)$$

Or:

$$[(\mathrm{E}+\mathrm{S})p^2+\mathrm{K}]\bar{y}_i(p)=\bar{F}_i(p) \tag{13a}$$

Where:

$\bar{y}_i(p)$ and $\bar{F}_i(p)$…Images of functions $q_j(t)$ and $F_j(t)$ for j = 4 up to 9;

p…Parameter of Laplace transform.

The solution of the system of linear algebraic equations (13) should be done by Cramer's rule.

$$\bar{y}_i(p)=\frac{D_j(p)}{D(p)}=\sum_{i=4}^{n=9}(-1)^{j+i}\,\bar{F}_i(p)\,\frac{D_{ji}(p)}{D(p)} \tag{14}$$

Where:

$D(p)$…Determinant of matrix of equations system (13) and is equal to:

$$D(p)=C_A\sum_{i=0}^{n}A_{2(n-1)}p^{2(n-1)} \qquad \text{for } n=9 \qquad C_A=1\text{-}s_{12}s_{21} \tag{15}$$

Equations for calculation of the real coefficients $A_{2(n-i)}$ were derived in [7-10], in general, these equations are valid for $n \geq 2$ and $0 \leq i < n$.

$$A_{2(n-i)}=1 \qquad A_{2(n-1)}=C_A^{1}\left\{\sum_{i=1}^{n}a_{ii}\text{-}s_{12}a_{21}\text{-}\,s_{21}a_{12}\text{-}s_{12}s_{21}\sum_{i=3}^{n}a_{ii}\right\}$$

$$A_{2(n-2)}=C_A^{1}\left\{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\begin{bmatrix}a_{ii} & a_{ij}\\ a_{ji} & a_{jj}\end{bmatrix}\text{-}\sum_{i=3}^{n}\left(\begin{bmatrix}a_{21} & a_{2i}\\ a_{i1} & a_{ii}\end{bmatrix}+s_{21}\begin{bmatrix}a_{12} & a_{1i}\\ a_{i2} & a_{ii}\end{bmatrix}\right)\text{-}s_{12}s_{21}\sum_{i=3}^{n-1}\sum_{j=i+1}^{n}\begin{bmatrix}a_{ii} & a_{ij}\\ a_{ji} & a_{jj}\end{bmatrix}\right\}$$

$$\begin{aligned}A_{2(n-3)}=C_A^{1}\Bigg\{&\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{l=j+1}^{n}\begin{bmatrix}a_{ii} & a_{ij} & a_{il}\\ a_{ji} & a_{jj} & a_{jl}\\ a_{li} & a_{lj} & a_{ll}\end{bmatrix}\text{-}s_{12}\sum_{i=3}^{n-1}\sum_{j=i+1}^{n}\begin{bmatrix}a_{21} & a_{2i} & a_{2j}\\ a_{i1} & a_{ii} & a_{ij}\\ a_{j1} & a_{ji} & a_{jj}\end{bmatrix}\text{-}s_{21}\sum_{i=3}^{n-1}\sum_{j=i+1}^{n}\begin{bmatrix}a_{12} & a_{1i} & a_{1j}\\ a_{i2} & a_{ii} & a_{ij}\\ a_{j2} & a_{ji} & a_{jj}\end{bmatrix}\text{-}\\ &\text{-}s_{12}s_{21}\sum_{i=3}^{n-2}\sum_{j=i+1}^{n-1}\sum_{l=j+1}^{n}\begin{bmatrix}a_{ii} & a_{ij} & a_{il}\\ a_{ji} & a_{jj} & a_{jl}\\ a_{li} & a_{lj} & a_{ll}\end{bmatrix}\Bigg\}\end{aligned} \tag{16}$$

$$\begin{aligned}A_{2(n-4)}=C_A^{1}\Bigg\{&\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{l=j+1}^{n-1}\sum_{k=l+1}^{n}det\,A_{ijkl}\text{-}\sum_{i=3}^{n-2}\sum_{j=i+1}^{n-1}\sum_{l=j+1}^{n}[s_{12}\,det\,A_{21}+s_{21}\,det\,A_{12}]\text{-}\\ &\text{-}s_{12}s_{21}\sum_{i=3}^{n-3}\sum_{j=i+1}^{n-2}\sum_{l=j+1}^{n-1}\sum_{k=l+1}^{n}det\,A_{ijkl}\Bigg\}\end{aligned}$$

Where:

$$det\,A_{ijkl}=\begin{bmatrix}a_{ii} & a_{ij} & a_{il} & a_{ik}\\ a_{ji} & a_{jj} & a_{jl} & a_{jk}\\ a_{li} & a_{lj} & a_{ll} & a_{lk}\\ a_{ki} & a_{ki} & a_{kl} & a_{kk}\end{bmatrix} \qquad det\,A_{21}=\begin{bmatrix}a_{21} & a_{2i} & a_{2j} & a_{2l}\\ a_{i1} & a_{ii} & a_{ij} & a_{il}\\ a_{j1} & a_{ji} & a_{jj} & a_{jl}\\ a_{l1} & a_{li} & a_{lj} & a_{ll}\end{bmatrix} \qquad det\,A_{21}=\begin{bmatrix}a_{12} & a_{1i} & a_{1j} & a_{1l}\\ a_{i2} & a_{ii} & a_{ij} & a_{il}\\ a_{j2} & a_{ji} & a_{jj} & a_{jl}\\ a_{l2} & a_{li} & a_{lj} & a_{ll}\end{bmatrix}$$

$$A_0=C_A^{1}det\,K$$

Calculation of coefficients $A_{2(n-i)}$ for $2 < i < n$ is hard to solve. To calculate the values of these coefficients it is advisable to use one of the calculation programs, e.g. MAPLE or MATLAB.

Determinant $D_j(p)$ in equation (14) is obtained substitution j-th column of determinant $D(p)$ by vector $\bar{F}_i(p)$ for i = 4 up to 9 from right side of equation (13). The determinant $D_{ji}(p)$ is the algebraic complement of the determinant $D_j(p)$ according to the i-th element of the j-th column of the vector $\bar{F}_i(p)$.

By adjusting the division between determinants $D_{ji}(p)$ and $D(p)$ to determine demanded functions $q_j(t)$ using back transformation of the images $\bar{y}_j(p)$. The theorem about image convolution can be utilized when the product of the image $\bar{F}_i(p)$ and transformed expression is determined.

The function in equation (15) with real coefficients (16) can be substituted by the product of quadratic binomials.

$$\sum_{i=0}^{n}A_{2(n-1)}p^{2(n-1)}=\prod_{i=1}^{n}(p^2+\omega_1^2) \tag{17a}$$

When the product on the right-hand side of (17a) is solved, we get the polynomial:

$$\sum_{i=0}^{n}A_{2(n-1)}p^{2(n-1)}=\sum_{i=0}^{n}B_{2(n-i)}p^{2(n-i)} \tag{17b}$$

Coefficients $B_{2(n-i)}$ can be solved as follows:

$$B_{2n}=1 \qquad B_{2(n\text{-}1)}=\sum_{i=1}^{n}\omega_i^2 \qquad B_{2(n\text{-}2)}=\sum_{i=1}^{n\text{-}1}\sum_{j=i+1}^{n}\omega_i^2\omega_j^2$$
$$B_{2(n\text{-}3)}=\sum_{i=1}^{n\text{-}2}\sum_{j=i+1}^{n\text{-}1}\sum_{l=j+1}^{n}\omega_i^2\omega_j^2\omega_l^2 \qquad B_{2(n\text{-}4)}=\sum_{i=1}^{n\text{-}3}\sum_{j=i+1}^{n\text{-}2}\sum_{l=j+1}^{n\text{-}1}\sum_{k=l+1}^{n}\omega_i^2\omega_j^2\omega_l^2\omega_k^2 \tag{18}$$
$$B_{2(n\text{-}n)}=B_0=\prod_{i=1}^{n}\omega_i^2$$

A system of equations to asses ω_i, ω_j, ω_l, … can be assembled by coefficients comparison of polynomials with the same exponents owing the parameter $p^{2(n-i)}$ located at the both sides of the equations (17a, b) $A_{2(n\text{-}i)} = B_{2(n\text{-}i)}$.

The simple equations are reached when equations (18) is compared with the relations describing the coupling of the coefficients of the algebraic equation and its root factors.

$$f(\omega^2)=\omega^{2n}\text{-}A_{2(n\text{-}1)}\omega^{2(n\text{-}1)}+A_{2(n\text{-}2)}\omega^{2(n\text{-}2)}\text{-}A_{2(n\text{-}3)}\omega^{2(n\text{-}3)}+(\text{-}1)^{n\text{-}1}A_2\omega^2+(\text{-}1)^nA_0=0 \tag{19}$$

Where:

ω_2…Angular frequency of functions $y_j(t)$, for $j = 1, 2, 3, \ldots, n$.

The most challenging aspect of the proposed procedure is calculating the eigenvalues of the frequency equation (19) using an appropriate numerical method. It is not only due to the need for numerical precision but also because it involves solution of equations (13). The right-hand part of equation (14) is defined by the solution of (13).

Since the polynomial of the determinant $D_{ji}(p)$ has non-zero coefficients only at even exponents of the parameter p in the solved case of undamped oscillation, and since the ratio of the determinants of (14) is a purely rational fractional function, the ratio (14) can be decomposed into a sum of partial fractions.

$$\frac{D_{ji}(p)}{D(p)}=\sum_{k=1}^{n}\frac{L_{ji,k}}{p^2+\omega_k^2} \tag{20}$$

Where the indeterminate coefficient, the constant $L_{ji,k}$ belongs to k-th of the partial fraction of the polynomial (20).

To determine these indeterminate coefficients for the i-th element in the sum (20), the determinant $D_{ji}(p)$ must be expressed as a polynomial.

$$D_{ji}(p)=\sum_{s=1}^{n}C_{ji,2(n\text{-}s)}p^{2(n\text{-}s)} \tag{21}$$

The sum of the partial fractions on the right hand side of equation (20) can be rewritten.

$$\sum_{k=1}^{n}\frac{L_{ji,k}}{p^2+\omega_k^2}=\frac{1}{\prod_{i=1}^{n}(p^2+\omega_i^2)}\Bigg[L_{ji,1}\prod_{s=2}^{n}(p^2+\omega_s^2)+L_{ji,2}\prod_{\substack{s=1\\ s\neq 2}}^{n}(p^2+\omega_s^2)+\ldots+ \\ +L_{ji,k}\prod_{\substack{s=1\\ s\neq k}}^{n}(p^2+\omega_k^2)+\ldots+L_{ji,n}\prod_{s=1}^{n\text{-}1}(p^2+\omega_s^2)\Bigg] \tag{22}$$

The individual products on the right hand side is given by:

$$L_{ji,k}=\prod_{\substack{s=1\\ s\neq k}}^{n}(p^2+\omega_s^2)=L_{ji,k}\sum_{\substack{p=1\\ p\neq k}}^{n}E_{k,2(n\text{-}s)}p^{2(n\text{-}s)} \tag{23}$$

Where the coefficients of the polynomial of the right-hand side of equation (23) are:

$$E_{k,2(n\text{-}1)}=1 \qquad E_{k,2(n\text{-}2)}=\sum_{\substack{p=1\\ p\neq k}}^{n}\omega_p^2 \qquad E_{k,2(n\text{-}3)}=\sum_{\substack{p=1\\ p\neq k}}^{n\text{-}1}\sum_{\substack{q=p+1\\ q\neq k}}^{n}\omega_p^2$$
$$E_{k,2(n\text{-}4)}=\sum_{\substack{p=1\\ p\neq k}}^{n\text{-}2}\sum_{\substack{q=p+1\\ q\neq k}}^{n\text{-}1}\sum_{\substack{r=q+1\\ r\neq k}}^{n}\omega_p^2\omega_q^2\omega_r^2 \qquad E_{k,2(n\text{-}n)}=\prod_{\substack{p=1\\ p\neq k}}^{n}\omega_p^2 \tag{24}$$

Substituting (21) and (22) into equation (20), taking into account (24), we get:

$$\sum_{s=1}^{n}C_{ji,2(n\text{-}s)}p^{2(n\text{-}s)}=\sum_{k=1}^{n}L_{ji,k}\sum_{s=1}^{n}E_{k,2(n\text{-}s)}p^{2(n\text{-}s)} \tag{25}$$

And comparing the coefficients at the same exponents of the parameter p in the polynomial of the numerators of both fractions (20), a system of n-linear algebraic equations for the calculation of the unknown uncertain coefficients $L_{ij,k}$ is obtained.

$$\sum_{k=1}^{n} L_{ji,k} E_{k,2(n-s)} = C_{ji,2(n-s)} \quad \text{for } s = 1, 2, \ldots, n \tag{26}$$

Where:

The index $j = 1 \div n$…A part of the quantities y_j;

The index $i = 4 \div n$…The excitation function vector $\overline{F}_i(p)$ appropriate elements;

The index $k = 1 \div n$…The partial fraction.

The results of the decomposition of the ratio of determinants (20) is partial fraction.

Equation (26) is expressed in matrix form

$$\begin{bmatrix} E_{1,2(n-1)} & E_{2,2(n-1)} & \cdots & E_{k,2(n-1)} & \cdots & E_{n,2(n-1)} \\ E_{1,2(n-2)} & E_{2,2(n-2)} & \cdots & E_{k,2(n-2)} & \cdots & E_{n,2(n-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{1,2(n-k)} & E_{2,2(n-k)} & \cdots & E_{k,2(n-k)} & \cdots & E_{n,2(n-k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{1,0} & E_{2,0} & \cdots & E_{k,0} & \cdots & E_{n,0} \end{bmatrix} \cdot \begin{bmatrix} L_{ji,1} \\ L_{ji,2} \\ \vdots \\ L_{ji,k} \\ \vdots \\ L_{ji,n} \end{bmatrix} = \begin{bmatrix} C_{ji,2(n-1)} \\ C_{ji,2(n-2)} \\ \vdots \\ C_{ji,2(n-k)} \\ \vdots \\ C_{ji,0} \end{bmatrix}$$

where the factors $E_{k,2(n-k)}$ are given by (23). After determining the constants $L_{ji,k}$ and substituting them into (20) and (14), we obtain the relation for the calculation of the image $\overline{y}_j(p)$

$$\overline{y}_j(p) = \sum_{i=4}^{n} (-1)^{j+i}\, \overline{F}_i(p) \sum_{k=1}^{n} \frac{L_{ji,k}}{p^2 + \omega_k^2}$$

After back-transformation we get the convolution integral.

$$q_j(t) = \sum_{i=4}^{n} (-1)^{j+i} \sum_{k=1}^{n} \frac{L_{ji,k}}{\omega_k} \int_0^t F_i(\tau) \sin \omega_k (t-\tau)\, d\tau \tag{27}$$

Where:

ω_k…The result of formula (19).

When the unknowns $q_j(t)$ are calculated as a results of equations system (8), the variables in the system (6) are assessed. Now it is simple to achieve the vertical displacement at all points of chassis or frame.

4 Conclusion

The objective of the article was to create and verify a method for the analytical solution of the assessment of the influence of structural and operational asymmetries of various types at different kinematic excitation, steady and transient, compared to the motion of a symmetrically arranged vehicle system. On the basis of the derived relationships, the permissible level of asymmetry can be determined for a specific vehicle while maintaining the driving safety and stability of the vehicle. The presented calculation procedure, together with the derived equations for the vehicle chassis, enables the calculation of the forces acting on the wheels namely theirs intensity and time profile, assessing the extent and permissibility of deformation in single spring components and structural components, and establishing damping rate to ensure physiological comfort during the vehicle's motion (when transporting passengers).

The use of the acquired knowledge will allow to adjust the vehicle elements in such a way that in operation there is no loss of contact of the wheel with the roadway and thus no critical situations.

The proposed solution enables the creation of an algorithm for the calculation of vertical displacements at arbitrary points of the model of rolling stock chassis vehicles with a large range of variants of a particular structural arrangement.

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